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Some Properties of* - Quasihyponormal Operators

Nidaa Muraeh Atheab and Salim Dawood Mohsen

Abstract--- *In this paper, we introduce a new generalization for hyponormal operator which is *-quasihyponormal and discussion some important theorems of this concept, as well as, some properties of this operator have been given following this work. Finally, this work given other objective is solvability of some types of bounded linear operator call λ- commuting operator equation when has *-quasihyponormal operator and fined the value λ of these equations in a Hilbert space H.*

Keywords--- *Hilbert Spaces, Hyponormal Operators,*-Quasihyponormal Operators.*

I. INTRODUCTION

The first to consider the idea of hyponormal operators was P. R. Halmos [4] in (1950) . in (1972) Shila Devi [3] introduced the definition of quasihy ponormal operators, which is a generalization for hyponormal operators, and in (1975), N.C.Shah and I.H. Sheth [7] introduced some results for quasihyponormal operators and they studied the relations of quasihyponormal operators with normal and hyponormal operators. In (1995), Moo Sang Lee [6] proved some spectra theorems for quasihyponormal operators. In (2011), Young Min Han [5], given some properties for quasihyponormal operators and quasi-M-hyponormal operators.

This paper divided into two sections; in section one, we recall some basic definitions we will need in this paper, also we have given some properties, examples and theorem for quasihyponormal operators. In section two we introduce a study on *-quasihyponormal operators, and given the essential relation of *- quasihyponormal with other types of linear hyponormal operator in a Hilbert space H.

In the last, we have given the following theorem:

1. Theorem

Let S, T:H$\rightarrow$H be an operators on Hsuch that $S^*$and T are *- quasihyponormal operators, such that $ST = \lambda TS \neq 0$, $\lambda \in \mathbb{C}$ and $ST^* = \lambda T^* S$, then TS^* is *-quasihyponormal operator if and only if $|\lambda|=1$.

1. Preliminaries

Through this paper, H represents a Hilbert space and any operator on H is bounded linear operator.

1.1. Definition [1 , P. 154]

Let T : H$\rightarrow$H be a bounded linear operator on a Hilbert space H ,then T is said to be **normal operator** if $T^*T = TT^*$, that is: $<T^*T x, x> =< TT^* x , x>$,for all $x \in H$.

1.2. Definition [2]

Let T:H$\rightarrow$H be an operator on H, then T is said to be **hyponormal operator** if $T^*T \geq TT^*$, that is: $< T^*T x, x> \geq< TT^* x , x>$,for all $x \in H$.

Nidaa Muraeh Atheab, Wasit University, College of Education, Department of Mathematics. E-mail: Nadaaalbadry1981@yahoo.com
Salim Dawood Mohsen, AL-Mustansirya University, College of Education, Department of Mathematics.
E-mail: dr_salim2015@yahoo.com

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***1.3. Definition* [3]**

Let T:H→Hbe an operator on H, then T is said to be **quasihyponormal operator**if

$T^*(T^*T - TT^*)\, T \geq 0$, that is: $(T^*)^2T^2 \geq (T^*T)^2$.

***1.4. Remark* [7]**

Let T: H→H be an operator on H , then:

i. If T is hyponormal operator, then T is quasihyponormal.

ii. If T quasihyponormal operator, then T need not be hyponormal. To show this consider the following example

Let Hbe two-dimensional Hilbert space and **K** be direct sum of an infinite number of copies of H. Let T = $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, S = $\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$. Define an operator $A_{T,S}$ $(x_1, x_2,\ldots) = (0, Tx_1,\ldots,Tx_n, Sx_{n+1},\ldots)$, then the operator A $_{T,S}$ is quasihyponormal operator, but not hyponormal.

iii. If T invertible and quasihyponormal operator, then T is hyponormal.

1.5. Theorem

Let T:H→H be an operator onH, then:

i. T is quasihyponormal operator if and only if $\| T^*Tx\| \leq \|T\, Tx \|$, for all x ∈H.[7]

ii. If T is quasihyponormal operator , M subspace of Hand invariant under T ,then $T|_M$ is quasihyponormal .[6]

1.6. Remark

Let S, T: H→H be quasihyponormal operators onH ,then (S + T) not necessarily be quasihyponormal operator. To illustrate this we give the following example:

The operators T = $\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ and S = $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ are quasihyponormal operators, butS + T = $\begin{bmatrix} 4 & 2 \\ -2 & 0 \end{bmatrix}$ is not quasihyponormal operator, since

$$(S + T)^*(\,(S + T)^*(S + T) - (S + T)(S + T)^*)\,(S + T) = \begin{bmatrix} -25 & -64 \\ -64 & 0 \end{bmatrix}.$$

Now, we give the conditions that make Remark (1.6) true.

1.7. Theorem

Let S, T: H→H be quasihyponormal operators , then (S + T) is quasihyponormal if

$ST = TS = S^*T = TS^* = 0$.

Proof:

Suppose that S and T are quasihyponormal operators,

$((S + T)^*)^2(S + T)^2 = (S + T)^*(S + T)^*(S + T)(S + T)$

$= (S^* + T^*)(S^* + T^*)(S + T)(S + T)$

$= (S^*S^* + S^*T^* + T^*S^* + T^*T^*)(SS + ST + TS + TT)$

$= (S^*S^* + (TS)^* + (ST)^* + T^*T^*)(SS + ST + TS + TT)$

$= (S^*S^* + T^*T^*)(SS + TT)$

$= (S^*S^*)(SS) + (S^*S^*)(TT) + (T^*T^*)(SS) + (T^*T^*)(TT)$

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$= (S^*)^2S^2 + S^*(S^*T)T + T^*(T^*S)S + (T^*)^2T^2$

$= (S^*)^2S^2 + S^*(S^*T)T + T^*(S^*T)^*S + (T^*)^2T^2$

$= (S^*)^2S^2 + (T^*)^2T^2$

$\geq (S^*S)^2 + (T^*T)^2$ ……………………………………………………(1)

On the other hand

$((S + T)^*(S + T))^2 = (S + T)^*(S + T)(S + T)^*(S + T)$

$= (S^* + T^*)(S + T)(S^* + T^*)(S + T)$

$= (S^*S + S^*T + T^*S + T^*T)(S^*S + S^*T + T^*S + T^*T)$

$= (S^*S + S^*T + (S^*T)^* + T^*T)(S^*S + S^*T + (S^*T)^* + T^*T)$

$= (S^*S + T^*T)(S^*S + T^*T)$

$= (S^*S)^2 + (S^*ST^*T) + (T^*TS^*S) + (T^*T)^2$

$= (S^*S)^2 + S^*(ST^*)T + T^*(TS^*)S + (T^*T)^2$

$= (S^*S)^2 + S^*(TS^*)^*T + T^*(TS^*)S + (T^*T)^2$

$= (S^*S)^2 + (T^*T)^2$ …………………………………………………(2)

By (1) and (2), we get $((S + T)^*)^2(S + T)^2 \geq ((S + T)^*(S + T))^2$

Therefore, (S + T) is quasihyponormal operator.

1.8. Remark

Let S, T : $H \to H$ be quasihyponormal operators onH ,then (S T) not necessarily be quasihyponormal operator. To explain this, we introduce the following example:

The operators $T = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ are quasihyponormal operators, but $ST = \begin{bmatrix} 3 & -2 \\ -6 & -1 \end{bmatrix}$ is not quasihyponormal operator, since

$$(S\,T)^*((ST)^*(ST) - (ST)(ST)^*)(ST) = \begin{bmatrix} -1440 & -240 \\ -240 & 160 \end{bmatrix}.$$

1.9. Theorem : [7]

Let S, T : $H \to H$ be quasihyponormal operators , then (ST) is quasihyponormal if
$ST=TS$ and $ST^* = T^*S$.

2. Main Result

2.1. Definition

Let T: $H \to H$ be an operator on H,then T is said to be **$*$-quasihyponormal** if $T(T^*T-TT^*)T^* \geq 0$, that is $(T\,T^*)^2 \geq T^2(T^*)^2$.To illustrate this definition consider the following example.

2.2. Examples

i. The operator $T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is *-quasihyponormal .Since $T(T^*T-TT^*)T^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

ii. The operator $S = \begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix}$ is not *-quasihyponormal .Since $S(S^*S - SS^*)S^* = \begin{bmatrix} 364 & 78 \\ 78 & -104 \end{bmatrix}$.

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2.3. Theorem

Let T: H→H be an operator on H, then T is *-quasihyponormal operator if and only if

$\|T^*T^* x\| \leq \| T T^* x\|$, for all x ∈H.

Proof:

Suppose that T is *-quasihyponormal operator.

For any x ∈H,

$$\|T^*T^* x\|^2 = < T^*T^* x, T^*T^* x >$$
$$= < x, T T T^*T^* x >$$
$$= < x, T^2 (T^*)^2 x >$$
$$\leq < x, (T T^*)^2 x >$$
$$= < x, T T^* T T^* x >$$
$$= < T T^* x, T T^* x >$$
$$= \| T T^* x\|^2$$

Hence , $\|T^* T^* x\|^2 \leq \| T T^* x\|^2$.

Now, Suppose that for any x ∈H

$$\|T^* T^* x\|^2 \leq \| T \ T^* x\|^2$$
$$< T^* T^* x, T^* T^* x > \leq < T T^* x, \quad T T^* x >$$
$$< x, T T T^* T^* x > \leq < x, \quad T T^* T T^* x >$$
$$< x, (T T^*)^2 - T^2 (T^*)^2 x > \geq 0$$

Therefore, T is *- quasihyponormal operator.■

2.4. Remarks and Examples

Let T : H→H be an operator on H, then :

i. If T is hyponormal operator, then T is *-quasihyponormal,

Proof:

Let x ∈H,

$$\|T T^* x\|^2 = < T T^* x, T T^* x >$$
$$= < x, (T T^*)^* T T^* x >$$
$$= < x, T T^* T T^* x >$$
$$\geq < x, T T T^* T^* x >$$
$$= < T^*T^* x, T^*T^* x >$$
$$= \|T^*T^* x\|^2$$

Therefore, T is *-quasihyponormal operator.

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ii. If T is *-quasihyponormal operator, then T is not necessary be hyponormal. To show this consider the following example

Let H be two-dimensional Hilbert space and K be direct sum of an infinite number of copies of H. Let T = $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ **, S =** $\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$**. Define an operator** $A_{T,S}$ $(x_1, x_2,\ldots) = (0, Tx_1,\ldots,Tx_n, Sx_{n+1},\ldots)$ **on K , then the operator** $A_{T,S}$ **is *- quasihyponormal operator, but not hyponormal.**

iii. If T is invertible *-quasihyponormal operator, then T is hyponormal.

Proof :

Since T invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$

Since T is *-quasihyponormal, then

$(TT^*)^2 \geq T^2(T^*)^2$

$TT^* \geq (TT^*)^{-1}TTT^*T^*$

$TT^* \geq (T^*)^{-1}T^{-1}TTT^*T^*$

$TT^* \geq (T^*)^{-1}\, I\, TT^*T^*$

$TT^* \geq (T^*)^{-1}TT^*T^*$

$I \geq (T^*)^{-1}TT^*T^*(TT^*)^{-1}$

$I \geq (T^*)^{-1}TT^*T^*(T^*)^{-1}T^{-1}$

$I \geq (T^*)^{-1}TT^*IT^{-1}$

$I \geq (T^*)^{-1}TT^*T^{-1}$

$T^* \geq TT^*T^{-1}$

$T^*T \geq TT^*$

Thus, T is hyponormal operator.

iv. If T is *-quasihyponormal operator, then λT is * - quasihyponormal operator, for any $\lambda \in \mathbb{R}$.

Proof:

$$\|(\lambda T)^*(\lambda T)^*x\| = \|\bar{\lambda}\bar{\lambda}T^*T^*x\|$$

$$= \|\lambda\lambda T^*T^*x\|$$

$$\leq \|\lambda\lambda TT^*x\|$$

$$= \|(\lambda T)(\lambda T)^*x\|$$

Hence, λT is *- quasihyponormal operator.

v. $(T-\lambda I)$ is not *- quasihyponormal operator for any $\lambda \in \mathbb{C}\backslash\{0\}$ .To show this consider the following example:

The operator T = $\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}$ is *- quasihyponormal operator.

But if $\lambda = 1$, then $(T-\lambda I) = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$ is not *- quasihyponormal ,since

$(T-\lambda I)((T-\lambda I)^*(T-\lambda I)-(T-\lambda I)(T-\lambda I)^*)(T-\lambda I)^* = \begin{bmatrix} -8 & 12 \\ 12 & 34 \end{bmatrix}$.

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2.5. Remark

The quasihyponormal and *-quasihyponormal are disjoint concepts. To illustrate this consider the following examples:

i. The operator T = $\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ is quasihyponormal and *-quasihyponormal. Since

$$(T^*)^2T^2 - (T^*T)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } (TT^*)^2 - T^2(T^*)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

ii. The operator S = $\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$ is not quasihyponormal and not *-quasihyponormal. Since

$$(S^*)^2S^2 - (S^*S)^2 = \begin{bmatrix} -36 & -20 \\ -20 & 4 \end{bmatrix} \text{ and } (SS^*)^2 - S^2(S^*)^2 = \begin{bmatrix} 36 & 20 \\ 20 & 4 \end{bmatrix}.$$

iii. The operator W = $\begin{bmatrix} 5 & 1 \\ 0 & 0 \end{bmatrix}$ is *- quasihyponormal but not quasihyponormal. Since

$$(WW^*)^2 - W^2(W^*)^2 = \begin{bmatrix} 51 & 0 \\ 0 & 0 \end{bmatrix}, \text{ but} (W^*)^2W^2 - (W^*W)^2 = \begin{bmatrix} 0 & -25 \\ -25 & -1 \end{bmatrix}.$$

2.6. Proposition

Let T : H→H be an *-quasihyponormal operator and M closed subspace of H. If M invariant under T ,then $T|_M$ is *- quasihyponormal operator.

Proof:

Suppose that T is *- quasihyponormal operator, and $T_1 = T|_M$, then: $Tx = T_1x$,for all $x \in M$.

Let $x \in M$, then

$$\|(T_1)^*(T_1)^* x\| = \|T^*T^* x\|$$
$$\leq \|T T^* x\|$$
$$\leq \|T_1 (T_1)^* x\|$$

Hence, $T|_M$ is *- quasihyponormal operator.

2.7. Remark

Let S,T : H→H be *- quasihyponormal operators on H, then (S+T) is not necessarily be *- quasihyponormal operator. To explain this consider the following example:

The operators T = $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, S = $\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$ are *- quasihyponormal operators.

But (S+T) = $\begin{bmatrix} 5 & 3 \\ -1 & 5 \end{bmatrix}$ is not *- quasihyponormal. Since,

$(S + T)((S + T)^*(S + T) - (S + T) (S + T)^*) (S + T)^* = \begin{bmatrix} -128 & 160 \\ 160 & 192 \end{bmatrix}$.

In the following theorem we will provide the conditions that make Remark (2.7) correct.

2.8. Theorem

Let S,T : H→H be *- quasihyponormal operators on aHsuch that $ST = TS = TS^* = S^*T = 0$, (zero mapping on H), then (S+T) is *- quasihyponormal operator.

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Proof:

Suppose that S and T are *- quasihyponormal operators

For any x $\in$H, then

$((S + T)(S + T)^*)^2 =((S + T)(S^* + T^*))^2$

$= (SS^* + ST^* + TS^* + TT^*)^2$

$=(SS^* + ST^*+(ST^*)^*+ TT^*)^2$

$=(SS^* + TT^*)^2$

$=(SS^* + TT^*)(SS^* + TT^*)$

$= (SS^*)^2 + (TT^*)^2$

$\geq S^2(S^*)^2 + T^2(T^*)^2$……………….(1)

On the other hand

$(S + T)^2 ((S + T)^*)^2 = (S + T)^2 (S^* + T^*)^2$

$= (S^2 + ST + TS + T^2)((S^*)^2 + S^*T^* + T^*S^* + (T^*)^2)$

$= (S^2 + T^2) ((S^*)^2 + (T^*)^2)$

$= S^2(S^*)^2 + S^2(T^*)^2 + T^2(S^*)^2 T^2(T^*)^2$

$= S^2(S^*)^2 + T^2(T^*)^2$………………(2)

By (1) and (2), we get (S+T) is *- quasihyponormal operator.

2.9. Remark

Let S,T :H$\rightarrow$Hbe *- quasihyponormal operators on H, then (ST) is not necessarily be *- quasihyponormal operator. To clarify this consider the following example:

The operators T = $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, S = $\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$ are *- quasihyponormal operators .

But (ST) =$\begin{bmatrix} 8 & 7 \\ -1 & 4 \end{bmatrix}$ is not *- quasihyponormal operator. Since

$(ST)((ST)^*(ST) – (ST)(ST)^*)(ST)^*$= $\begin{bmatrix} 2864 & 2528 \\ 2528 & 464 \end{bmatrix}$, and the determent $\begin{vmatrix} 2864 & 2528 \\ 2528 & 464 \end{vmatrix}$ = -5061888

Now, the following theorem give the conditions that make Remark(2.9) is true

2.10. Theorem

Let S,T : H$\rightarrow$H be *-quasihyponormal operators on Hsuch that$(ST)^2 = S^2 T^2$ and $STT^* = TT^*S$, then (ST) is *-quasihyponormal operator.

Proof:

Suppose that S, T are *- quasihyponormal operators and x $\in$H, then

$$\begin{aligned} \|(ST)(ST)^*x\| &= \|STT^*S^*x\| \\ &= \|TT^*SS^*x\| \\ &\geq \|T^*T^*S^*S^*x\| \\ &= \|(T^*)^2(S^*)^2x\| \end{aligned}$$

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$$= \| (T^2)^* (S^2)^* x \|$$
$$= \|((S^2 T^2)^* x \|$$
$$= \| ((S T)^2)^* x \|$$
$$= \|((S T)^*)^2 x \|$$
$$= \| (ST)^* (S T)^* x \|$$

Therefore, (ST) is *- quasihyponormal operator.

In the following theorem we solve the equation ST = λTS, where S and T are *- quasihyponormal operators.

2.11. Theorem

Let S, T: H→H be operators onH, such that S^* and T are *- quasihyponormal operators, ST = λTS ≠ 0, λ∈ℂ and $ST^*=\lambda T^*S$, then TS^* is *-quasihyponormal if and only if |λ|=1.

Proof:

Suppose that S^* and T are *- quasihyponormal operators.

For any x ∈H,

$$\|((T S^*)(T S^*)^* x \| = \|T S^* S T^* x \|$$
$$\geq \| T S S T^* x \|$$
$$= \left\|\frac{1}{\lambda^2} S S T T^* x\right\|$$
$$\geq \left\|\frac{1}{\lambda^2} S S T^* T^* x\right\|$$
$$= \left\|\frac{\lambda}{\lambda^2} S T^* S T^* x\right\|$$
$$= \left\|\frac{1}{\lambda} (T S^*)^* (T S^*)^* x\right\|$$
$$= \frac{1}{|\lambda|} \|(T S^*)^* (T S^*)^* x \|$$

Hence, $\|((T S^*)(T S^*)^* x \| \geq \frac{1}{|\lambda|} \|(T S^*)^* (T S^*)^* x \|$

If |λ| = 1, then $\|((T S^*)(T S^*)^* x \| \geq \|(T S^*)^* (T S^*)^* x \|$ and $TS^*$is *-quasihyponormal.

On the other hand,if TS^* is *-quasihyponormal ,

then by Definition(2.1), we have$\| ((T S^*)(T S^*)^* x \| \geq \|(T S^*)^* (T S^*)^* x \|$.

But $\|((T S^*)(T S^*)^* x \| \geq \frac{1}{|\lambda|} \|(T S^*)^* (T S^*)^* x \|$.

Therefore, |λ| = 1

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