Three-Step Iterative Method with Fifth Order Convergence for Solving Non-linear Equations

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Abstract--- *In this paper, a new three step iterative method is proposed based on the Newton's Method to obtain the numerical solution of non linear equation. We prove that our method take over fifth order convergence and the efficiency of the recommended method is shown by the numerical examples comparing with existing Method.*

Keywords--- *Iterative Method, Nonlinear Equation, Newton's Method, Convergence Analysis.*

I. INTRODUCTION

To solve transcendental equation we have so many methods in the literature of numerical analysis. These days much attention has given to develop different iterative methods to find the root of non-linear equation. One of the well known, very important methods to find the root of non linear equation is Newton's method

$$f(x) = 0 \tag{1.1}$$

where f is a scalar function on open interval D is a classical Newton's method (NR) given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 0,1,2,\ldots \tag{1.2}$$

This quadratically convergent method and its efficiency index is $\sqrt{2} = 1.414$.

An efficient Newton type method with fifth order of convergence for solving nonlinear equations (FANG) proposed by Liang Fang [3] is given by

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= y_n - \frac{5f'^2(x_n)+3f'^2(y_n)}{f'^2(x_n)+7f'^2(y_n)} \cdot \frac{f(y_n)}{f'(x_n)} \end{aligned}\right\} \tag{1.3}$$

A New third order iterative method for solving non linear equations (CH) proposed by C. Chun [1] is given by

$$\left.\begin{aligned} z_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - \frac{1}{2}\left(3 - \frac{f'(z_n)}{f'(x_n)}\right)\frac{f(x_n)}{f'(x_n)} \end{aligned}\right\} \tag{1.4}$$

A Composite third order Newton- Steffensen's method for solving nonlinear equations (SM) proposed by J. R. Sharma [2] is given by

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DOI: 10.37200/IJPR/V24I5/PR201696
Received: 07 Feb 2020 | Revised: 01 Mar 2020 | Accepted: 18 Mar 2020

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - \frac{(f(x_n))^2}{f'(x_n)(f(x_n) - f(y_n))} \end{aligned}\right\} \quad (1.5)$$

An iterative method for solving nonlinear equations (ZANG) proposed by Zhonyong [7] is given by

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= y_n - \left(1+\left(\frac{f(y_n)}{f(x_n)}\right)^2\right)\frac{f(y_n)}{f'(y_n)} \end{aligned}\right\} \quad (1.6)$$

In this paper we define a new three step iterative method which raises the convergence and increases the efficiency of a function and establish the convergence of this approach. In the next section different numerical tests confirm the theoretical results and allow us to compare these variants with classical methods.

II. FIFTH ORDER CONVERGENT METHOD

A fifth order iterative method to solving nonlinear equations proposed by M. Rafiullah[5] given by

$$x_{n+1} = z_n - \frac{[f(y_n)]^2 f''(y_n)}{2[f'(x_n)]^2 f'(y_n)} \quad (2.1)$$

where $z_n = y_n - \frac{f(y_n)}{f'(y_n)}$ and $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$.

The order of convergence of this method is 5 and the efficiency index is 3.18.

Replacing z_n by y_n and y_n by x_n in “(2.1)” , we obtain

$$x_{n+1} = y_n - \mu \frac{f(x_n) f'(x_n)}{2[f'(x_n)]^2} \quad (2.2)$$

where $\mu = \frac{f(x_n) f''(x_n)}{[f'(x_n)]^2}$

Algorithm

For a given x_0 we compute x_{n+1} by the iterative scheme

DOI: 10.37200/IJPR/V24I5/PR201696
Received: 07 Feb 2020 | Revised: 01 Mar 2020 | Accepted: 18 Mar 2020

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n + \left(f'(y_n) - f'(x_n)\right)\left(\frac{f(x_n)}{2\left(f'(x_n)^2\right)}\right) \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(y_n)} \end{aligned}\right\} \tag{2.5}$$

The method "(2.5)" is called as fifth order convergent method (MMS), requires 2 functional evaluations and 2 of its first derivatives.

III. Convergence Criteria

Thoerem: Let $x_0 \in D$ be a single zero of a sufficiently differentiable function f for an open interval D. If x_0 is in the neighborhood of x^*. Then "(2.5)" has fifth order convergence.

Proof: Let x^* be a single zero of "(1.1)" and

$$x^* = x_n + \varepsilon_n$$

then $f\left(x^*\right) = 0$ .Expanding $f\left(x^*\right)$ by Taylor's series about x_n , we have

$$f(x_n) = f'\left(x^*\right)\left(\varepsilon_n + c_2\varepsilon_n^2 + c_3\varepsilon_n^3 + c_4\varepsilon_n^4 + \ldots\right) \tag{3.1}$$

$$f'(x_n) = f'\left(x^*\right)\left(1 + 2c_2\varepsilon_n + c_3\varepsilon_n^2 + 4c_4\varepsilon_n^3 + \ldots\right) \tag{3.2}$$

Dividing "(3.1)" by "(3.2)",

we have

$$\frac{f(x_n)}{f'(x_n)} = \begin{pmatrix} \varepsilon_n - c_2\varepsilon_n^2 - \left(2c_3 - 2c_2^2\right)\varepsilon_n^3 - \\ \left(3c_4 - 7c_2c_3 + 4c_2^3\right)\varepsilon_n^4 + \ldots \end{pmatrix} \tag{3.3}$$

From the first step of MMS, we obtain

$$y_n = x^* + \alpha_n$$

where

$$\alpha_n = c_2\varepsilon_n^2 + \left(2c_3 - 2c_2^2\right)\varepsilon_n^3 + \left(3c_4 - 7c_2c_3 + 4c_2^3\right)\varepsilon_n^4 + \ldots$$

Now,

DOI: 10.37200/IJPR/V24I5/PR201696
Received: 07 Feb 2020 | Revised: 01 Mar 2020 | Accepted: 18 Mar 2020

$$f'(y_n) = f'(x^*)\left(1+2c_2^2\varepsilon_n^2+\left(4c_2c_3-4c_2^3\right)\varepsilon_n^3+\dots\right)$$

Then, we have $f'(y_n)-f'(x_n) = f'(x^*)(-2c_2\varepsilon_n+\left(2c_2^2-3c_3\right)\varepsilon_n^2+$

$\left(4c_2c_3-4c_2^3-4c_4\right)\varepsilon_n^3+\left(6c_2c_4-14c_2^2c_3+8c_2^4+3c_2^2c_3\right)\varepsilon_n^4+..$ and

$$\left(f'(y_n)-f'(x_n)\right)\left[\frac{f(x_n)}{2\left(f'(x_n)^2\right)}\right] = c_2\varepsilon_n^2+\left(8c_2^2-\frac{3}{2}c_3\right)\varepsilon_n^3$$

$$+\left(-57c_2^3+\frac{47}{2}c_2c_3-2c_4\right)\varepsilon_n^3+\dots \tag{3.4}$$

From the second step of "(2.5)", we get

$$z_n = x^*+\rho$$

Where $\rho=\left(6c_2^2-\frac{1}{2}c_3\right)\varepsilon_n^3+\left(-53c_2^3+\frac{33}{2}c_2c_3+c_4\right)\varepsilon_n^4+\dots$

Now

$$\frac{f(z_n)}{f'(y_n)}=\left(6c_2^2-\frac{1}{2}c_3\right)\varepsilon_n^3+\left(-53c_2^3+\frac{33}{2}c_2c_3+c_4\right)\varepsilon_n^4+\frac{1}{2}\left(64c_2c_4-16c_2^4-405c_2^2c_3\right)\varepsilon_n^5+\dots \tag{3.5}$$

Finally, by the third step of "(2.5)", we have

$$x_{n+1}=x^*+\frac{1}{2}\left(64c_2c_4-16c_2^4-405c_2^2c_3\right)\varepsilon_n^5+\dots.$$

Hence this method is fifth order convergence and its efficiency index is $\sqrt[4]{5}=1.4953$

IV. NUMERICAL EXAMPLES

After consider the some examples considered by Jayakumar [6] and Noor [4] compared our method (MMS) with NR, FANG, CH, SM and ZANG methods. The computations are carried out by using mpmath-PYTHON software programming and comparison of number of iterations for these methods are obtained such that $|x_{n+1}-x_n|<10^{-201}$ and $|f(x_{n+1})|<10^{-201}$.

The test functions and simple zeros are given below $f_1(x)=\sin(2\cos x)-1-x^2+e^{\sin\left(x^3\right)}$, $x^*=-0.78489$

$$f_2(x)=xe^{\left(x^2\right)}-\sin^2 x+3\cos x+5,\ x^*=-1.20764$$

$$f_3(x)=\sin x+\cos x+x,\ x^*=-0.4566$$

DOI: 10.37200/IJPR/V24I5/PR201696
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$$f_4(x) = (x+2)e^{x} - 1,\ x^* = -0.44285$$

$$f_5(x) = x^2 + \sin\left(\frac{x}{5}\right) - \frac{1}{4},\ x^* = 0.40999$$

Table IV (a): Comparison of Efficiency Index

Methods	P	N	EI
NR	2	2	1.414
CH	6	4	1.442
SM	3	3	1.442
FANG	5	4	1.495
ZANG	5	4	1.495
MMS	5	4	1.495

where P is order of the convergence, N is the number of functional values per iteration and EI is the Efficiency Index.

Table IV (b): Comparison of Different Methods

f	Method	x_0	*n*	*er*	*fv*
	NR	-0.6	10	1.6(201)	4.1(201)
	FANG		5	3.2(201)	4.1(201)
f_1	CH		7	8.9(201)	2.4(200)
	SM		DIVEREGENT		
	ZANG		5	7.3(201)	2.4(200)
	MMS		5	3.2(201)	4.1(201)
	NR	-1.2	8	6.8(200)	1.3(198)
	FANG		4	3.7(200)	6.4(199)
f_2	CH		6	3.1(200)	6.4(200)
	SM		6	9.7(201)	6.4(199)
	ZANG		4	9.4(200)	6.4(199)
	MMS		4	3.7(200)	1.3(198)
	NR	-0.5	8	7.7(201)	1.8(200)
	FANG		4	4.8 (201)	5.3(201)
f_3	CH		6	2.4(201)	5.3(201)
	SM		DIVERGENT		
	ZANG		4	6.9(201)	5.3(201)
	MMS		4	5.3(201)	1.8(200)
	NR	-0.7	10	6.9(201)	1.1(200)
	FANG		5	4.4(201)	4.1 (201)
f_4	CH		7	2.4(201)	4.1(201)
	SM		7	5.3(201)	1.1(200)
	ZANG		5	5.9(200)	4.1(201)
	MMS		5	4.8(201)	4.1(201)
	NR	1.5	11	2.1(201)	2.2(201)
	FANG		6	5.7(201)	2.2(201)
f_5	CH		8	2.1(201)	2.2(201)
	SM		7	4.1(202)	2.2(201)
	ZANG		6	9.6(200)	2.2(201)
	MMS		6	5.7 (201)	2.2(201)

f	Method	x_0	*n*	*er*	*fv*
	NR	-1.3	10	1.6(201)	4.1(201)
	FANG		5	3.2(201)	4.1(201)
f_1	CH		7	1.6(201)	4.1(201)
	SM		7	7.3(201)	2.4(201)
	ZANG		5	4.9(201)	4.1(201)
	MMS		5	3.2(201)	4.1(201)
	NR	1	12	3.1(200)	6.1(199)
	FANG		6	3.7(200)	6.4(199)
f_2	CH	DIVERGENT			
	SM	DIVERGENT			
	ZANG		7	3.1(200)	1.3(198)
	MMS		6	6.2(200)	6.4(199)
	NR	1.25	12	7.7(201)	1.8(200)
	FANG		6	5.3(201)	5.3(201)
f_3	CH		7	7.7(201)	1.8(201)
	SM		7	6.1(201)	1.8(200)
	ZANG	DIVERGENT			
	MMS		5	4.8 (201)	5.3(201)
	NR	0.4	11	2.4(201)	4.1(201)
	FANG		6	4.8 (201)	4.1(201)
f_4	CH		8	2.4(201)	4.1(201)
	SM	DIVERGENT			
	ZANG		6	4.1(201)	1.1(200)
	MMS		6	4.9(201)	4.1(201)
	NR	1.8	11	7.7(201)	7.7(201)
	FANG		6	5.7(201)	7.7(201)
f_5	CH		8	2.1(201)	2.2(201)
	SM		6	5.7(201)	7.7(201)
	ZANG		6	6.5(201)	2.2(201)
	MMS		6	5.7(201)	2.2(201)

Where x_0 is the initial approximation, *n* is the number of iterations, *er* is the error and *fv* is the functional value.

DOI: 10.37200/IJPR/V24I5/PR201696
Received: 07 Feb 2020 | Revised: 01 Mar 2020 | Accepted: 18 Mar 2020

V. Conclusion

Then, we In this method we developed new fifth order convergent method with efficiency index 1.495. It requires two functional evaluations and two of its first derivatives. Table IV(a) compares the efficiency of different methods and the computational results in table IV(b) exhibit the superiority of the new method over the NR, FANG, CH, NK, SM and ZANG, FOCM methods in terms of number of iterations and accuracy.

References

[1] C. Chun, K, Yong-II, "Several new third-order iterative methods for solving nonlinear equations", *Acta Appl. Math*., 109 (2010), pp. 1053-1063.

[2] J.R. Sharma, "A composite third order newton-steffensen method for solving nonlinear equations", *Appl. Math. Comput*, 169 (2005), pp. 242-246.

[3] Ling Fang, Lisun, Guoping He, "An efficient Newton type method with fifth order convergence for solving nonlinear equations". *Computational and applied mathematics*, Vol 27, N. 3 (2008), pp. 269–274.

[4] M.A. Noor, W.A. Khan, K.I. Noor, E, Al-said, "Higher-order iterative methods free from second order derivative for solving non linear equations", *International journal of physical Sciences*,6(8) (2011), pp. 1887-1893.

[5] M. Rafiullah Proposed "A fifth order iterative method to solving nonlinear equations". *Numerical Analysis and Applications*, Vol. 4, No. 3 (2011), pp. 239-243.

[6] S. Parimila, M. Kalyanasundaram and J. Jayakumar. " Modified Ostrowski Method For Solving Nonlinear Equation". *International Journal of pure and applied Mathematics*, vol. 120, No. 8 (2018), pp. 59-67.

[7] Zhongyonh Hu, Liu Guocai and Li Tian, "An iterative method with ninth order convergence for solving Nonlinear equations". *Int. J. Contemp. Math. Sciences*, Vol. 6 no. 1 (2011) , pp. 17-23.

DOI: 10.37200/IJPR/V24I5/PR201696
Received: 07 Feb 2020 | Revised: 01 Mar 2020 | Accepted: 18 Mar 2020