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Fixed Point Theorems Via Best Approximation

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Abstract
In this paper, we prove some common fixed point theorems of Gregus type in Banach spaces and give application of our fixed point theorems to best approximation theory. Our work generalizes several earlier results on fixed points in this direction.

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Key words: *Common fixed point, compatible mappings, weakly compatible mappings, best approximant.*

1 INTRODUCTION:

Several interesting results using fixed point theory are given in approximation theory. During the last 130 years or so this area has attracted the attention of several mathematicians.
The fundamental result in the best approximation theory was given by Meinaradus [11], afterwards in (1969), Brosowski [1] theorem has been a basic important result, many authors have studied the applications of fixed point theorem to best approximation theory. Subrahmanyan [21], S.P. Singh [18], M.L. Singh [18], Carbone ([2] [3]), Sahab, Khan and Sessa [14], Hicks and Humphries [15], In (1988), Sahab, Khan and Sessa [14] generalized the result of Singh [19], Recently Pathak, Cho-kang [13] gave an applications of Jungck's [9], fixed point theorem to best approximation theory they extended the result of Singh [19] and Sahab et. al. [14].

In this paper, we prove some common fixed point theorems of Gregus type for compatible mappings and weakly compatible mappings in Banach spaces and give application of our fixed point theorems to best approximation theory.

2 PRELIMINARIES:

The following definitions and results will be used in this paper.

2.1 Definition :
Let C be a subset of normed linear space X. Then

(i) A mapping $T : X \to X$ is said to be **contractive** on X. if $\|Tx - Ty\| \le \|x - y\|$ for all *x, y* in *X* (resp. *C*)

The set $D_a$of best **(*C, a*)- approximants** to $\bar{x}$ consists of the point y in C such that $a\|y - \bar{x}\| = \inf\{\|z - \bar{x}\| : z \in C\}$, where $\bar{x}$ in a point of X, then for $0 < a \le 1$.

(ii) Let *D* denote the set of best **C-approximants** to $\bar{x}$. for *a* = 1, our definition reduces to the Set D of best C-approximants to $\bar{x}$.

(iii) A subset *C* of *X* is said to be **star-shaped** with respect to a point $q \in C$ if, for all *x* in *C* and all $\lambda \in [0,1]$, $\lambda x + (1-\lambda)q \in C$,where the point q is called the **star-centre** of C.

(iv) A**convex set** is star-shaped with respect to each of its points, but not conversely. For an example, the set $C = \{0\} \times [0,1] \cup [1,0] \times \{0\}$ is star-shaped with respect to $(0,0) \in C$ as the star-centre of C, but it is not convex.

Throughout this chapter F (T) denotes the set of fixed points of T on X.
By relaxing the linearity of the operator T and convexity of Din the original statement of Brosowski [1],Singh [19]proved the following results.

2.2. Theorem :
Let C be a T-invariant subset of a normed linear space *X*. Let $T : C \to C$ be a contractive operator on C and let $\bar{x} \in F(T)$. if $D \subseteq X$ is non empty, compact and starshaped, then $D \cap F(T) \neq \phi$.

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In the subsequent paper Singh (19), observed that only non expansiveness of T on $D' = D \cup \{\bar{x}\}$ is necessary. Further, Hicks and Humphries (7) have shown the assumption $T : C \to C$ can be weakened to the condition $T : \partial C \to C$ $y \in C,$ i.e., $y \in D$ is not neceassarily in the interior of *C*, where ∂C denotes the boundary of *C*.

Recently, Sahab, Khan and Sessa [14]generalized Theorem 2.2 as in the following.

2.3 Theorem:
Let *X* be a Banach space let $T, I : X \to X$ be operators and C be a subset of X such that $T : \partial C \to C$ and $\bar{x} \in F(T) \cap F(I)$. Further, suppose that T and I satisfy

$$\|Tx - Ty\| \le \|Ix - Iy\|$$

for all x, y in D', I is linear, continuous on *D* and *ITx* = *TIx* for *x* in *D* if D is non empty, compact and starshaped with respect to a point $q \in F(I)$ and $I(D) = D,$ then $D \cap F(T) \cap F(I) \ne \phi$.

In [8], Jungck defined the concept of compatibility of two mappings, which includes weakly commuting mappings Sessa [15] as proper sub class.

2.4 Definition :
Let *X* be a normed linear space and let $S, T : X \to X$ be two mappings *S* and *T* are said to be compatible if, whenever $\{x_n\}$ is a sequence in *X* such that $Sx_n, Tx_n \to x \in X$, then

$$\|STx_n - TSx_n\| \to 0 \text{ as } n \to \infty$$

In (1998), Jungck and Rhoades[10] introduced the notion of weakly compatible maps and showed that compatible maps are weakly compatible but converse need not be true.

2.5 Definition :
A pair of *S* and *T* is called **weakly compatible** pair if they commute at coincidence points.

2.6 Example :
Consider *X* = [0,2] with the usual metric *d*. Define mappings $S, T : X \to X$ by
$Sx = 0$ if $x = 0$, $Sx = 0.15$ if $x > 0$
$Tx = 0$ if $x = 0$, $Tx = 0.3$ if $0 < x \le 0.5$, $Tx = x - 0.35$ if $x > 0.5$

Since *S* and *T* commute at coincidence point $0 \in X,$ so *S* and *T* are weakly compatible maps to see that *S* and *T* are not compatible, let us consider a decreasing sequence $\{x_n\}$ where $x_n = 0.5 + \left(\frac{1}{n}\right), n = 1,2,...$ Then $Sx_n \to 0.15, Tx_n \to 0.15$ but $STx_n \to 0.15, TSx_n \to 0.3$ as $n \to \infty$. Thus weakly compatible compatible maps need not be compatible.

3. MAIN RESULTS :
First of all, we prove a common fixed point theorem of Gregus type for compatible mappings in Banach space. Our Theorem is improvement of results of Gregus [6], Jungck [9], Sharma and Deshpande [17].
Throughout this section, we assume that *X* is Banach space and *C* is non empty closed convex subset of *X*.
Now, we prove our main theorem.

3.1 Theorem :
Let *S* and *T* be compatible mappings of *C* into itself satisfying the following condition:

$$\begin{aligned}\|Tx - Ty\| \le{}& a\|Sx - Sy\| + b\max\{\|Tx - Sx\|, \|Ty - Sy\|\} \\ &+ c\max\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|\} \\ &+ d\max\left\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|, \frac{1}{2}\left(\|Ty - Sx\| + \|Tx - Sy\|\right)\right\} \qquad \text{... (1)}\end{aligned}$$

for all *x, y* in *C* where *a,b,c,d>0, a+b+c+d*=1 and *a+c+d<* $\sqrt{a}$ if *S* is linear and continuous in *C* and $T(C) \subset S(C)$. Then *T* and *S* have a unique common fixed point *z* in *C* and *T* is continuous at *z*.

Proof :
Consider $x = x_0$ be an arbitarary point in *C* and choose points x_1, x_2 and x_3 in *C* such that
$Sx_1 = Tx$, $Sx_2 = Tx_1$, $Sx_3 = Tx_2$
This can be done since $T(C) \subset S(C)$. for r = 1,2,3,... (1) leads to

$$\begin{aligned}\|Tx_r - Sx_r\| &= \|Tx_r - Tx_{r-1}\| \\ &\le a\|Sx_r - Sx_{r-1}\| + b\max\{\|Tx_r - Sx_r\|, \|Tx_{r-1} - Sx_{r-1}\|\} \\ &+ c\max\{\|Sx_r - Sx_{r-1}\|, \|Tx_r - Sx_r\|, \|Tx_{r-1} - Sx_{r-1}\|\} \\ &+ d\max\{\|Sx_r - Sx_{r-1}\|, \|Tx_r - Sx_r\|, \|Tx_{r-1} - Sx_{r-1}\|, \frac{1}{2}(\|Tx_{r-1} - Sx_r\| + \|Tx_r - Sx_{r-1}\|)\}\end{aligned}$$

which shows that,since

$$\|Sx_r - Sx_{r-1}\| = \|Tx_{r-1} - Sx_{r-1}\|,$$

we have, for *r* = 1,2,3,...

$$\|Tx_r - Sx_r\| \le \|Tx_{r-1} - Sx_{r-1}\|. \qquad ....(2)$$

From (1) and (2) we have

$$\begin{aligned}\|Tx_2 - Sx_1\| &= \|Tx_2 - Tx\| \\ &\le a\|Sx_2 - Sx\| + b\max\{\|Tx_2 - Sx_2\|, \|Tx - Sx\|\} \\ &+ c\max\{\|Sx_2 - Sx\|, \|Tx_2 - Sx_2\|, \|Tx - Sx\|\} \\ &+ d\max\{\|Sx_2 - Sx\|, \|Tx_2 - Sx_2\|, \|Tx - Sx\|, \frac{1}{2}(\|Tx - Sx_2\| + \|Tx_2 - Sx\|)\} \\ &\le a\|Tx_1 - Sx\| + b\max\{\|Tx - Sx\|, \|Tx - Sx\|\} \\ &+ c\max\{\|Tx_1 - Sx\|, \|Tx - Sx\|, \|Tx - Sx\|\} \\ &+ d\max\{\|Tx_1 - Sx\|, \|Tx - Sx\|, \|Tx - Sx\|, \frac{1}{2}(\|Tx - Tx_1\| + \|Tx_2 - Sx\|)\} \\ &\le 2a\|Tx - Sx\| + b\|Tx - Sx\| + 2c\|Tx - Sx\| + d\max\{\|Tx_1 - Sx\|, \|Tx - Sx\|, \|Tx - Sx\|, \\ &\quad \frac{1}{2}(\|Tx - Sx_1\| + \|Tx_1 - Sx_1\| + \|Tx_2 - Sx_2\| + \|Sx_2 - Sx\|)\} \\ &\le 2a\|Tx - Sx\| + b\|Tx - Sx\| + 2c\|Tx - Sx\| + d\max\{\|Tx_1 - Sx\|, \|Tx - Sx\|, \|Tx - Sx\|, \\ &\quad \frac{1}{2}(\|Tx - Sx\| + \|Tx - Sx\| + \|Tx_1 - Sx\|)\} \\ &\le 2a\|Tx - Sx\| + b\|Tx - Sx\| + 2c\|Tx - Sx\| + 2d\|Tx - Sx\| \\ &\le \{(2a + 2c + 2d) + b\}\|Tx - Sx\| \qquad ....(3)\end{aligned}$$

We shall now define a point

$$z = \left(\frac{1}{2}\right)x_2 + \left(\frac{1}{2}\right)x_3.$$

Since *C* is convex, $z \in C$ and *S* being linear

$$\begin{aligned}Sz &= \left(\frac{1}{2}\right)Sx_2 + \left(\frac{1}{2}\right)Sx_3 \\ &= \left(\frac{1}{2}\right)Tx_1 + \left(\frac{1}{2}\right)Tx_2 \qquad ....(4)\end{aligned}$$

It follows from (2), (3) and (4) that

$$\|Sz - Sx_1\| = \left\|\left(\frac{1}{2}\right)Tx_1 + \left(\frac{1}{2}\right)Tx_2 - Sx_1\right\|$$

$$\leq \left(\frac{1}{2}\right)\|Tx_1 - Sx_1\| + \left(\frac{1}{2}\right)\|Tx_2 - Sx_1\|$$

$$\leq \left(\frac{1}{2}\right)\|Tx - Sx\| + \left(\frac{1}{2}\right)\{(2a + 2c + 2d) + b\}\|Tx - Sx\|$$

$$\leq \left(\frac{1}{2}\right)\{1 + (2a + 2c + 2d) + b\}\|Tx - Sx\| \qquad ....(5)$$

By (2) and (4), we have

$$\|Sz - Sx_2\| = \left\|\left(\frac{1}{2}\right)Tx_1 + \left(\frac{1}{2}\right)Tx_2 - Sx_2\right\|$$

$$\leq \left(\frac{1}{2}\right)\|Tx_2 - Sx_2\|$$

$$\leq \left(\frac{1}{2}\right)\|Tx - Sx\|. \qquad ...(6)$$

By (1) and (6) we have

$$\|Tz - Sz\| = \left\|Tz - \left(\frac{1}{2}\right)Tx_1 - \left(\frac{1}{2}\right)Tx_2\right\|$$

$$\leq \left(\frac{1}{2}\right)\|Tz - Tx_1\| + \left(\frac{1}{2}\right)\|Tz - Tx_2\|$$

$$\leq \left(\frac{1}{2}\right)a\|Sz - Sx_1\| + \left(\frac{1}{2}\right)b\max\{\|Tz - Sz\|, \|Tx_1 - Sx_1\|\}$$

$$+ \left(\frac{1}{2}\right)c\max\{\|Sz - Sx_1\|, \|Tz - Sz\|, \|Tx_1 - Sx_1\|\}$$

$$+ \left(\frac{1}{2}\right)d\max\{\|Sz - Sx_1\|, \|Tz - Sz\|, \|Tx_1 - Sx_1\|,$$

$$\frac{1}{2}(\|Tx_1 - Sz\| + \|Tz - Sx_1\|)\} + \left(\frac{1}{2}\right)a\|Sz - Sx_2\| + \left(\frac{1}{2}\right)b\max\{\|Tz - Sz\|, \|Tx_2 - Sx_2\|\}$$

$$+ \left(\frac{1}{2}\right)c\max\{\|Sz - Sx_2\|, \|Tz - Sz\|, \|Tx_2 - Sx_2\|\}$$

$$+ \left(\frac{1}{2}\right)d\max\{\|Sz - Sx_2\|, \|Tz - Sz\|, \|Tx_2 - Sx_2\|,$$

$$\frac{1}{2}(\|Tx_2 - Sz\| + \|Tz - Sx_2\|)\}$$

$$\leq \left(\frac{1}{4}\right)a[1 + 2a + 2c + 2d + b]\|Tx - Sx\| + \left(\frac{1}{2}\right)b\max\{\|Tz - Sz\|, \|Tx - Sx\|\}$$

$$+ \left(\frac{1}{2}\right)c\max\left\{\frac{1}{2}(1 + 2a + 2c + b + 2d)\|Tx - Sx\|, \|Tz - Sz\|, \|Tx - Sx\|\right\}$$

$$+ \left(\frac{1}{2}\right)d\max\left[\frac{1}{2}(1 + 2a + 2c + b + 2d)\|Tx - Sx\|, \|Tz - Sz\|, \|Tx - Sx\|,\right.$$

$$\frac{1}{2}\{(2+2a+2c+2d+b)\|Tx-Sx\|+\|Tz-Sz\|\}\Big]+\left(\frac{1}{4}\right)a\|Tx-Sx\|$$

$$+\left(\frac{1}{2}\right)b\max\{\|Tz-Sz\|,\|Tx-Sx\|\}+\left(\frac{1}{2}\right)c\max\left\{\frac{1}{2}\|Tx-Sx\|,\|Tz-Sz\|,\|Tx-Sx\|\right\}$$

$$+\left(\frac{1}{2}\right)d\max\left\{\frac{1}{2}\|Tx-Sx\|,\|Tz-Sz\|,\|Tx-Sx\|,\ \frac{1}{2}\left(2\|Tx-Sx\|+\|Tz-Sz\|\right)\right\}$$

$$\leq \lambda\|Tx-Sx\| \qquad \text{….(7)}$$

where

$$\lambda=\left(\frac{1}{4}\right)a[2+2a+2c+2d+b]+b+\frac{1}{4}c[1+2a+2c+b+2d]$$

$$+\frac{1}{2}c+\frac{1}{4}d[2+2a+2c+b+2d]+d$$

$$<\frac{1}{4}a(3+\sqrt{a})+\frac{1}{4}c(2+\sqrt{a})+b+\frac{1}{2}c+\frac{d}{4}(3+\sqrt{a})+d$$

$$<\frac{a}{4}+\frac{3a}{4}+b+c+d$$

$$=a+b+c+d=1$$

So we have 0< λ <1.

Since$x$ is an arbitary point in C, from (7), it follows that there exists a sequence $\{z_n\}$ in C such that

$$\|Tz_0-Sz_0\|\leq\lambda\|Tx_0-Sx_0\|,$$

$$\|Tz_1-Sz_1\|\leq\lambda\|Tz_0-Sz_0\|,$$

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$$\|Tz_n-Sz_n\|\leq\lambda\|Tz_{n-1}-Sz_{n-1}\|,$$

which yield that

$$\|Tz_n-Sz_n\|\leq\lambda^{n+1}\|Tx_0-Sx_0\|,$$

and so we have

$$\lim_{n\to\infty}\|Tz_n-Sz_n\|=0 \qquad \text{….(8)}$$

Setting $K_n=\left\{x\in C:\|Tx-Sx\|\leq\frac{1}{n}\right\}$

for n = 1,2, … then (8) shows that

$K_n\neq\phi$ for n = 1,2,…

and $K_1\supset K_2\supset K_3\supset\ldots$

obviously, we have $\overline{TK_n}\neq\phi$ and

$\overline{TK_n}\supset\overline{TK_{n+1}}$ for n = 1,2,…

for any x, y in K_n by (1), we have

$$
\begin{aligned}
\|Tx-Ty\| &\le a\|Sx-Sy\| + n^{-1}b + c\max\{\|Sx-Sy\|, n^{-1}\} \\
&\quad + d\max\{\|Sx-Sy\|, n^{-1}, \tfrac{1}{2}(n^{-1}+\|Sx-Sy\|+n^{-1}+\|Sx-Sy\|)\} \\
&\le a\|Sx-Sy\| + n^{-1}b + c\max\{\|Sx-Sy\|, n^{-1}\} \\
&\quad + d\max\{\|Sx-Sy\|, n^{-1}, (n^{-1}+\|Sx-Sy\|)\} \\
&\le a(2n^{-1}+\|Tx-Ty\|) + n^{-1}b + c(2n^{-1}+\|Tx-Ty\|) + d(3n^{-1}+\|Tx-Ty\|) \\
&= [a+c]2n^{-1} + [a+c+d]\|Tx-Ty\| + n^{-1}b + 3n^{-1}d
\end{aligned}
$$

Therefore,

$$\|Tx-Ty\| \le n^{-1}\{2[a+c]+b+3d\}(1-a-c-d)^{-1}$$

Thus we have

$$\lim_{n\to\infty} diam\left(\overline{TK_n}\right) = \lim_{n\to\infty} diam\left(TK_n\right) = 0$$

By Cantor's theorem, there exists a point u in C such that

$$\bigcap_{n=1}^{\infty}\left(\overline{TK_n}\right) = \{u\}.$$

Since $u \in C$ for each $n = 1,2,\ldots$ there exists a point y_n in TK_n such that

$$\|y_n - u\| < n^{-1}$$

Then there exists a point x_n is K_n such that

$$\|u - Tx_n\| < n^{-1}$$

and so $Tx_n \to u \;\; as \;\; n \to \infty$.

Since $x_n \in k_n$, we have also

$$\|Tx_n - Sx_n\| < n^{-1}$$

and so $Sx_n \to u \;\; as \;\; n \to \infty$.

Since S is continuous $STx_n \to Su \; and \; SSx_n \to Su \;\; as \;\; n \to \infty$.

Moreover $\|TSx_n - STx_n\| \to 0 \; as \; n \to \infty$.

Since S and T are compatible and $Tx_n \to Sx_n \to u \;\; as \;\; n \to \infty$, we have $TSx_n \to Su$.

By (1), we have

$$
\begin{aligned}
\|Tu-Su\| &\le \|Tu-TSx_n\| + \|TSx_n - Su\| \\
&\le a\|Su-SSx_n\| + b\max\{\|Tu-Su\|, \|TSx_n - SSx_n\|\} \\
&\quad + c\max\{\|Su-SSx_n\|, \|Tu-Su\|, \|TSx_n-SSx_n\|\} \\
&\quad + d\max\{\|Su-SSx_n\|, \|Tu-Su\|, \|TSx_n-SSx_n\|, \\
&\quad \tfrac{1}{2}(\|TSx_n-Su\|+\|Tu-SSx_n\|)\} + \|TSx_n-Su\|
\end{aligned}
$$

Letting $n \to \infty$, we obtain

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$$
\begin{aligned}
\|Tu - Su\| &\le a\|Su - Su\| + b\max\{\|Tu - Su\|, \|Su - Su\|\} \\
&\quad + c\max\{\|Su - Su\|, \|Tu - Su\|, \|Su - Su\|\} \\
&\quad + d\max\{\|Su - Su\|, \|Tu - Su\|, \|Su - Su\|, \\
&\quad \frac{1}{2}(\|Su - Su\| + \|Tu - Su\|)\} + \|Su - Su\| \\
&= (b + c + d)\|Tu - Su\| \\
&= (1 - a)\ \|Tu - Su\|.
\end{aligned}
$$

So we have $Tu = Su$.

Thus $TSu = STu$ and $TTu = TSu = STu$ since S and T are compatible. Furthermore, we have

$$
\begin{aligned}
\|TTu - Tu\| &\le a\|STu - Su\| + b\max\{\|TTu - STu\|, \|Tu - Su\|\} \\
&\quad + c\max\{\|STu - Su\|, \|TTu - STu\|, \|Tu - Su\|\} \\
&\quad + d\max\{\|STu - Su\|, \|TTu - STu\|, \|Tu - Su\| \\
&\quad \frac{1}{2}(\|Tu - STu\| + \|TTu - Su\|)\} \\
&= (a + c + d)\|TTu - Tu\|
\end{aligned}
$$

This leads to $\|TTu - Tu\| = 0$ since $(a + c + d) < \sqrt{a}$.

Let $z = Tu = Su$.

Then $Tz = z$ and $Sz = STz = TSz = Tz = z$.

Thus z is a unique common fixed point of T and S. The uniqueness of z is a consequence of inequality (1). Now, we show that T is continuous at z. Let $\{y_n\}$ be a sequence in C such that $y_n \to z$.

Since S is continuous, $Sy_n \to Sz$, By (1), we have

$$
\begin{aligned}
\|Ty_n - Tz\| &\le a\|Sy_n - Sz\| + b\max\{\|Ty_n - Sy_n\|, \|Tz - Sz\|\} \\
&\quad + c\max\{\|Sy_n - Sz\|, \|Ty_n - Sy_n\|, \|Tz - Sz\|\} \\
&\quad + d\max\{\|Sy_n - Sz\|, \|Ty_n - Sy_n\|, \|Tz - Sz\|, \frac{1}{2}(\|Tz - Sy_n\| + \|Ty_n - Sz\|)\} \\
&\le a\|Sy_n - Sz\| + b\max\{\|Ty_n - Tz\| + \|Tz - Sy_n\|\} \\
&\quad + c\max\{\|Sy_n - Sz\|, \|Ty_n - Tz\| + \|Tz - Sy_n\|\} \\
&\quad + d\max\{\|Sy_n - Sz\|, \|Ty_n - Tz\| + \|Tz - Sy_n\|, \|Tz - Sz\|, \\
&\quad \frac{1}{2}(\|Tz - Sy_n\| + \|Ty_n - Sz\|)\} \\
&\le a\|Sy_n - Sz\| + b\{\|Ty_n - Tz\| + \|Sz - Sy_n\|\} + c\{\|Ty_n - Tz\| + \|Sz - Sy_n\|\} \\
&\quad + d\{\|Ty_n - Tz\| + \|Sz - Sy_n\|, \\
\|Ty_n - Tz\| &\le (a + b + c + d)\|Sy_n - Sz\| + (b + c + d)\|Ty_n - Tz\| \\
&\le (a + b + c + d)(1 - b - c - d)^{-1}\|Sy_n - Sz\|
\end{aligned}
$$

Therefore, we have $Ty_n \to Tz$ and so T is continuous at z.

This completes the proof.

As a consequences of our Theorem 3.1, we have the following results.

3.2 Corallary:
Let S and T be compatible mappings of C into itself satisfying the following condition:

$$\begin{aligned}\|Tx-Ty\| \le a\|Sx-Sy\| + b\max\{\|Tx-Sx\|, \|Ty-Sy\|\} \\ + c\max\{\|Sx-Sy\|, \|Tx-Sx\|, \|Ty-Sy\|\}\end{aligned}$$

for all x, y in C where $a,b,c>0$, $a+b+c=1$ and $a+c<\sqrt{a}$ if S is linear and continuous in C and $T(C) \subset S(C)$. Then T and S have a unique common fixed point z in C and T is continuous at z.

Corallary 3.1.1 shows the result of Sharma and Deshpande [16], which obtain by putting $d = 0$.
Now if b=0, c=0 then we get the following corallary

3.3 Corallary:
Let S and T be compatible mappings of C into itself satisfying the following condition:

$$\|Tx-Ty\| \le a\|Sx-Sy\| + (1-a)\max\{\|Tx-Sx\|, \|Ty-Sy\|\}$$

for all x, y in C, 0 <a <1, if S is linear and continuous in C and $T(C) \subset S(C)$, Then T and S have a unique common fixed pointz in C and T is continuous at z.

3.4 Remark :
Corallary (3.3) also proves continuity of T, so it improves the result of Jungck[9].
if we put $a = b = c = 0$ then we get the following result

3.5 Corallary :
Let S and T be compatible mappings of C into itself satisfying the following condition:

$$\|Tx-Ty\| \le d\max\left\{\|Sx-Sy\|, \|Tx-Sx\|, \|Ty-Sy\|, \frac{1}{2}(\|Ty-Sx\| + \|Tx-Sy\|)\right\}$$

for all x, y in C where $0 \le d < 1$, if S is linear and continuous in C and $T(C) \subset S(C)$. Then T and S have a unique common fixed pointz in C and T is continuous at z.

To demonstrate the validity of our Theorem 3.1, we have the following example

3.6 Example :
Let $X = R$ *and* $C = [0,1]$ with the usual norm. Consider the mappings T and S on C defined as

$$Tx = \frac{1}{4}x \quad and \quad Sx = \frac{1}{2}x \quad for\ all\ x \in C$$

Then $T(C) = \left[0, \frac{1}{4}\right] \subset S(C) = \left[0, \frac{1}{2}\right]$.

It is easy to see that S is linear and continuous.
Further, T and S are compatible if $\lim_{n\to\infty} x_n = 0$, *where* $\{x_n\}$ is a sequence in C such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 0$ for some $0 \in C$.
If we take $a = 1/9, b = 13/18, c = 3/18, d = 0$ we see that the condition (1) of our Theorem 3.1, is satisfied also we have

$a+b+c=1$ and $a + c < \sqrt{a}$.

Thus all the conditions of Theorem 3.1 are satisfied and 0 is the unique common fixed point of S and T.
Now, in our next theorem, we give an application of our fixed point theorem to best approximation theory. We improve the results of Pathak, Cho-Kang [13]and Sharma-Deshpande[16].

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3.7 Theorem :
Let *T* and *S* be mapping of *X* into itself. Let $T:\partial C \to C$ and $\bar{x} \in F(T) \cap F(S)$. Further, suppose that *T* and *S* satisfy the condition (1), for all x, y in $D'_a = D_a \cup \{x\} \cup E$, where
$E = \{q \in X : Tx_n, Sx_n \to q, \{x_n\} \subset D_a\}$, *a,b,c,d>0, a+b+c+d*=1, *a+c+d*< $\sqrt{a}$, *S* is linear, continuous on D_a and *T,S* are compatible in D_a, if D_a is nonempty, compact, convex and $S(D_a)=D_a$, then $D_a \cap F(T) \cap F(S) \neq \phi$.

Proof :
Let $y \in D_a$ and hence *Sy* is in D_a since $S(D_a) = D_a$.
Further, if $y \in \partial C$, then *Ty* is in *C*, since $T(\partial C) \subseteq C$. from (1), it follows that

$$\|Ty-\bar{x}\| = \|Ty - T\bar{x}\| \le a\|Sy - S\bar{x}\| + bmax\{\|Ty-Sy\|, \|T\bar{x}-S\bar{x}\|\}$$
$$+cmax\{\|Sy-S\bar{x}\|, \|Ty-Sy\|, \|T\bar{x}-S\bar{x}\|\}$$
$$+dmax\left\{\|Sy-S\bar{x}\|, \|Ty-Sy\|, \|T\bar{x}-S\bar{x}\|, \frac{1}{2}(\|T\bar{x}-Sy\| + \|Ty-S\bar{x}\|)\right\}$$
$$\|Ty-\bar{x}\| \le a\|Sy-\bar{x}\| + b\max\{\|Ty-Sy\|, \|\bar{x}-\bar{x}\|\}$$
$$+c\max\{\|Sy-\bar{x}\|, \|Ty-Sy\|, \|\bar{x}-\bar{x}\|\}$$
$$+d\max\left\{\|Sy-\bar{x}\|, \|Ty-Sy\|, \|\bar{x}-\bar{x}\|, \frac{1}{2}(\|\bar{x}-Sy\| + \|Ty-\bar{x}\|)\right\}$$
$$\|Ty-\bar{x}\| \le a\|Sy-\bar{x}\| + b\{\|Ty-\bar{x}\| + \|\bar{x}-Sy\|\}$$
$$+c\max\{\|Sy-\bar{x}\|, \|Ty-\bar{x}\| + \|\bar{x}-Sy\|\}$$
$$+d\max\left\{\|Sy-\bar{x}\|, \|Ty-Sy\|, \frac{1}{2}(\|\bar{x}-Sy\| + \|Ty-\bar{x}\|)\right\}$$
$$\|Ty-\bar{x}\| \le a\|Sy-\bar{x}\| + b\{\|Ty-\bar{x}\| + \|\bar{x}-Sy\|\} + c\{\|Ty-\bar{x}\| + \|\bar{x}-Sy\|\}$$
$$+d\max\left\{\|Sy-\bar{x}\|, \|Ty-\bar{x}\| + \|\bar{x}-Sy\|, \frac{1}{2}(\|\bar{x}-Sy\| + \|Ty-\bar{x}\|)\right\}$$
$$\|Ty-\bar{x}\| \le a\|Sy-\bar{x}\| + b\|Ty-\bar{x}\| + b\|\bar{x}-Sy\| + c\|Ty-\bar{x}\| + c\|\bar{x}-Sy\|$$
$$+d\{\|Ty-\bar{x}\| + \|\bar{x}-Sy\|\}$$
$$\|Ty-\bar{x}\| \le (a+b+c+d)\|Sy-\bar{x}\| + (b+c+d)\|Ty-\bar{x}\|$$
$$(1-b-c-d)\|Ty-\bar{x}\| \le \|Sy-\bar{x}\|$$
$$a\|Ty-\bar{x}\| \le \|Sy-\bar{x}\|,$$

which implies $a\|Ty-\bar{x}\| \le \|Sy-\bar{x}\|$ and so *Ty* is in D_a. Thus *T* maps D_a into itself.

Proceeding as in Theorem 3.1, we can show that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u \qquad \text{….(9)}$$

Therefore, for a sequence $\{x_n\}$ in D_a the existence of (9) is guaranteed whenever $\mathrm{Da} \subset \mathrm{Kn}$. Moreover $u \in E$. Since *S* and *T* are compatible and *S* is continuous, we have

$$\lim_{n\to\infty} TSx_n = Su$$
and $$\lim_{n\to\infty} S^2 x_n = Su$$

By (1), we have

$$\begin{aligned}\|TSx_n - \bar{x}\| &= \|TSx_n - T\bar{x}\| \\ &\le a\|S^2x_n - S\bar{x}\| + b\max\{\|TSx_n - S^2x_n\|, \|T\bar{x} - S\bar{x}\|\} \\ &+ c\max\{\|S^2x_n - S\bar{x}\|, \|TSx_n - S^2x_n\|, \|T\bar{x} - S\bar{x}\|\} \\ &+ d\max\{\|S^2x_n - S\bar{x}\|, \|TSx_n - S^2x_n\|, \|T\bar{x} - S\bar{x}\| \\ &\frac{1}{2}(\|T\bar{x} - S^2x_n\| + \|TSx_n - S\bar{x}\|)\},\end{aligned}$$

which implies, as $n \to \infty$

$$\begin{aligned}\|Su - \bar{x}\| &\le a\|Su - \bar{x}\| + b\max\{\|Su - Su\|, \|\bar{x} - \bar{x}\|\} \\ &+ c\max\{\|Su - \bar{x}\|, \|Su - Su\|, \|\bar{x} - \bar{x}\|\} \\ &+ d\max\{\|Su - \bar{x}\|, \|Su - Su\|, \|\bar{x} - \bar{x}\|, \frac{1}{2}(\|\bar{x} - Su\| + \|Su - \bar{x}\|)\} \\ &\le a\|Su - \bar{x}\| + c\|Su - \bar{x}\| + d\|Su - \bar{x}\| \\ \|Su - \bar{x}\| &\le \sqrt{a}\ \|Su - \bar{x}\|.\end{aligned}$$

Hence $Su = \bar{x}$, By (1) again, we have

$$\begin{aligned}\|Tu - \bar{x}\| &= \|Tu - T\bar{x}\| \\ &\le a\|Su - S\bar{x}\| + b\max\{\|Tu - Su\|, \|T\bar{x} - S\bar{x}\|\} \\ &+ c\max\{\|Su - S\bar{x}\|, \|Tu - Su\|, \|T\bar{x} - S\bar{x}\|\} \\ &+ d\max\{\|Su - S\bar{x}\|, \|Tu - Su\|, \|T\bar{x} - S\bar{x}\|, \frac{1}{2}(\|T\bar{x} - Su\| + \|Tu - S\bar{x}\|)\}\end{aligned}$$

which gives, by taking $Su = \bar{x}$

$$\begin{aligned}\|Tu - \bar{x}\| &\le a\|\bar{x} - \bar{x}\| + b\max\{\|Tu - \bar{x}\|, \|\bar{x} - \bar{x}\|\} \\ &+ c\max\{\|\bar{x} - \bar{x}\|, \|Tu - \bar{x}\|, \|\bar{x} - \bar{x}\|\} \\ &+ d\max\{\|\bar{x} - \bar{x}\|, \|Tu - \bar{x}\|, \|\bar{x} - \bar{x}\|, \frac{1}{2}(\|\bar{x} - \bar{x}\| + \|Tu - \bar{x}\|)\} \\ &\le b\|Tu - \bar{x}\| + c\|Tu - \bar{x}\| + d\|Tu - \bar{x}\| \\ \|Tu - \bar{x}\| &\le (1-a)\|Tu - \bar{x}\|.\end{aligned}$$

So $Tu = \bar{x}$.
Next, we consider

$$\begin{aligned}\|Tu - Tx_n\| &\le a\|Su - Sx_n\| + b\max\{\|Tu - Su\|, \|Tx_n - Sx_n\|\} \\ &+ c\max\{\|Su - Sx_n\|, \|Tu - Su\|, \|Tx_n - Sx_n\|\} \\ &+ d\max\{\|Su - Sx_n\|, \|Tu - Su\|, \|Tx_n - Sx_n\|, \frac{1}{2}(\|Tx_n - Su\| + \|Tu - Sx_n\|)\}\end{aligned}$$

Letting $n \to \infty$, we get

$$\|\bar{x}-u\| \le a\|\bar{x}-u\| + b\max\{\|\bar{x}-\bar{x}\|, \|u-u\|\} + c\max\{\|\bar{x}-u\|, \|\bar{x}-\bar{x}\|, \|u-u\|\}$$
$$+ d\max\{\|\bar{x}-u\|, \|\bar{x}-\bar{x}\|, \|u-u\|, \frac{1}{2}(\|u-\bar{x}\| + \|\bar{x}-u\|)\}$$
$$\|\bar{x}-u\| \le (a+c+d)\ \|\bar{x}-u\|$$
$$\le \sqrt{a}\ \ \|\bar{x}-u\|, \text{ since } a+c+d < \sqrt{a}$$

and so $\bar{x} = u$, i.e., $u=Su=Tu$. By Theorem 3.1, u must be unique.

Hence E = {u}. Then $D_a^1 = D_a \cup \{u\}$

Let $\{k_n\}$ be a monotonically non-decreasing sequence of real numbers such that $0 \le k_n < 1$ and $\overline{\lim}_{n\to\infty} k_n = 1$. Let $\{x_j\}$be a sequence in D_a^1 satisfying (2), for each $n \in N$, define a mapping $T_n : D_a^1 \to D_a^1$ by

$T_n x_j = k_n T x_j + (1-k_n)p$

for each $n \in N$, it is possible to define such a mapping T_n. Since D_a' is starshaped with respect to $p \in F(S)$.
Since S is linear, we have

$$T_n S x_j = k_n T S x_j + (1-k_n)p,$$
$$S T_n S x_j = k_n S T x_j + (1-k_n)p.$$

By compatibility of S and T, we have for each $n \in N$

$$0 \le \lim_{j\to\infty} \|T_n S x_j - S T_n x_j\|$$
$$\le k_n \lim_{j\to\infty} \|TSx_j - STx_j\| + \lim_{j\to\infty} (1-k_n)\|p-p\|$$

and so

$$\lim_{j\to\infty} \|T_n S x_j - S T_n x_j\| = 0$$

whenever $\lim_{j\to\infty} Sx_j = \lim_{j\to\infty} T_n x_j = u$ since we have

$$\lim_{j\to\infty} T_n x_j = k_n \lim_{j\to\infty} Tx_j + (1-k_n)u$$
$$= k_n u + (1-k_n)u$$
$$= u.$$

Thus S and T_n are compatible on D_a' for each n and $T_n(D_a') \subset D_a' = S(D_a')$.
On the other hand by (1), for all $x, y \in D_a'$,we have for all $j \ge n$ and n fixed

$$\|T_n x - T_n y\| = k_n \|Tx - Ty\|$$
$$\le k_j \|Tx - Ty\|$$
$$< \|Tx - Ty\|$$
$$\le a\|Sx - Sy\| + b\max\{\|Tx - Sx\|, \|Ty - Sy\|\}$$
$$+ c\max\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|\}$$
$$+ d\max\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|, \frac{1}{2}(\|Ty - Sx\| + \|Tx - Sy\|)\}$$

$$\leq a\|Sx-Sy\|+b\max\{\|Tx-T_nx\|+\|T_nx-Sx\|,\|Ty-T_ny\|+\|T_ny-Sy\|\}$$
$$+c\max\{\|Sx-Sy\|,\|Tx-T_nx\|+\|T_nx-Sx\|,\|Ty-T_ny\|+\|T_ny-Sy\|\}$$
$$+d\max\{\|Sx-Sy\|,\|Tx-T_nx\|+\|T_nx-Sx\|,\|Ty-T_ny\|+\|T_ny-Sy\|,$$
$$\frac{1}{2}(\|Ty-T_ny\|+\|T_ny-Sx\|+\|Tx-T_nx\|+\|T_nx-Sy\|)\}$$
$$\leq a\|Sx-Sy\|+b\max\{(1-k_n)\|Tx-p\|+\|T_nx-Sx\|,(1-k_n)\|Ty-p\|+\|T_ny-Sy\|\}$$
$$+c\max\{\|Sx-Sy\|(1-k_n)\|Tx-p\|+\|T_nx-Sx\|,(1-k_n)\|Ty-p\|+\|T_ny-Sy\|\}$$
$$+d\max\{\|Sx-Sy\|,(1-k_n)\|Tx-p\|+\|T_nx-Sx\|,(1-k_n)\|Ty-p\|+\|T_ny-Sy\|$$
$$\frac{1}{2}((1-k_n)\|Ty-p\|+\|T_ny-Sx\|+(1-k_n)\|Tx-p\|+\|T_nx-Sy\|)\}$$

Hence for all $j \geq n$, we have

$$\|T_nx-T_ny\|\leq a\|Sx-Sy\|+b\max\{(1-k_j)\|Tx-p\|+\|T_nx-Sx\|,(1-k_j)\|Ty-p\|+\|T_ny-Sy\|\}$$
$$+c\max\{\|Sx-Sy\|(1-k_j)\|Tx-p\|+\|T_nx-Sx\|,(1-k_j)\|Ty-p\|+\|T_ny-Sy\|\}$$
$$+d\max\{\|Sx-Sy\|,(1-k_j)\|Tx-p\|+\|T_nx-Sx\|,(1-k_j)\|Ty-p\|+\|T_ny-Sy\|, \quad \text{....(10)}$$
$$\frac{1}{2}((1-k_j)\|Ty-p\|+\|T_ny-Sx\|+(1-k_j)\|Tx-p\|+\|T_nx-Sy\|)\}$$

Thus, since $\overline{\lim}_{j\to\infty} e_j = 1$, from (10), for every $n \in N$, we have

$$\|T_nx-T_ny\|\leq \overline{\lim}_{j\to\infty}\|T_nx-T_ny\|$$
$$<\overline{\lim}_{j\to\infty}\Big[a\|Sx-Sy\|+b\max\{(1-k_j)\|Tx-p\|+\|T_nx-Sx\|,(1-k_j)\|Ty-p\|+\|T_ny-Sy\|\}$$
$$+c\max\{\|Sx-Sy\|(1-k_j)\|Tx-p\|+\|T_nx(x)-Sx\|,(1-k_j)\|Ty-p\|+\|T_ny-Sy\|\}$$
$$+d\max\{\|Sx-Sy\|,(1-k_j)\|Tx-p\|+\|T_nx-Sx\|,(1-k_j)\|Ty-p\|+\|T_ny-Sy\|,$$
$$\frac{1}{2}((1-k_j)\|Ty-p\|+\|T_ny-Sx\|+(1-k_j)\|Tx-p\|+\|T_nx-Sy\|)\}\Big]$$

which implies

$$\|T_nx-T_ny\|=a\|Sx-Sy\|+b\max\{\|T_nx-Sx\|,\|T_ny-Sy\|\}$$
$$+c\max\{\|Sx-Sy\|,\|T_nx-Sx\|,\|T_ny-Sy\|\}$$
$$+d\max\{\|Sx-Sy\|,\|T_nx-Sx\|,\|T_ny-Sy\|,\frac{1}{2}(\|T_ny-Sx\|+\|T_nx-Sy\|)\}$$

for all $x.y \in D'_a$, therefore by Theorem 3.1 for every $n \in N$, T_n and S have a unique common fixed point x_n in D'_a, i.e., every $n \in N$, we have
$F(T_n)\cap F(S)=\{x_n\}$.

Now, the compactness of D_a ensures that $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ which converges to a point in $D_a$since

$$x_{n_i}=T_{n_i}x_{n_i}=k_{n_i}Tx_{n_i}+(1-k_{n_i})p \quad \text{....(11)}$$

and T is continuous, we have as $i \to \infty$ in (3.2.3) $z = Tz$ i.e., $z \in D_a \cap F(T)$.

Further, the continuity of *S* implies that

$$Sz = S(\lim_{i\to\infty} x_{n_i}) = \lim_{i\to\infty} Sx_{n_i} = \lim_{i\to\infty} x_{n_i} = z$$

i.e., $z \in F(S)$, therefore, we have $z \in D_a \cap F(T) \cap F(S)$ and so.

$$D_a \cap F(T) \cap F(S) \neq \phi$$

This completes the proof.
As a consequence of our Theorem 3.2, we have the following result.

3.8 Corallary :
Let *T* and *S* be mapping of *X* into itself. Let $T : \partial C \to C$ and $\bar{x} \in F(T) \cap F(S)$. Further, suppose that *T* and *S* satisfy.

$$\|Tx - Ty\| \leq a\|Sx - Sy\| + b\max\{\|Tx - Sx\|, \|Ty - Sy\|\}$$

$$+ c\max\left\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|\}, \frac{1}{2}(\|Ty - Sx\| + \|Tx - Sy\|)\right\}$$

for all x, y in $D'_a = D_a \cup \{x\} \cup E$, where

$E = \{q \in X : Tx_n, Sx_n \to q, \{x_n\} \subset D\}$, *a,b,c>0, a+b+c=1, a+c<* $\sqrt{a}$, *S* is linear, continuous on D_a and *T,S* are compatible in D_a, if D_a is nonempty, compact, convex and $S(D_a)=D_a$, then $D_a \cap F(T) \cap F(S) \neq \phi$.

This corollary is obtained by putting d = 0 in Theorem 3.2 and is the result of Sharma and Deshpande[16]*d* = 0.
Now, we obtain the following result due to Pathak, Cho and Kang[13] by putting *c* = *d*= 0 in Theorem 3.7.

3.9 Corallary :
Let T and S be mappings of X into itself. Let $T : \partial C \to C$ and $\bar{x} \in F(T) \cap F(S)$. Further, suppose that T and S satisfy.

$$\|Tx - Ty\| \leq a\|Sx - Sy\| + (1-a)\max\{\|Tx - Sx\|, \|Ty - Sy\|\}$$

for all x, y in $D'_a = D_a \cup \{x\} \cup E$, where

$E = \{q \in X : Tx_n, Sx_n \to q, \{x_n\} \subset D\}$, $0 < a < 1$, if *S* is linear, continuous on D_a and *T,S* are compatible in D_a, if D_a is nonempty, compact, convex and $S(D_a)=D_a$, then $D_a \cap F(T) \cap F(S) \neq \phi$.

Now, by putting *a* = *b* = *c* = 0 in Theorem 3.2 we get the following result.

3.10 Corallary :
Let *T* and *S* be mapping of *X* into itself. Let $T : \partial C \to C$ and $\bar{x} \in F(T) \cap F(S)$. Further, suppose that *T* and *S* satisfy

$$\|Tx - Ty\| \leq d\max\left\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|, \frac{1}{2}(\|Ty - Sx\| + \|Tx - Sy\|)\right\}$$

for all *x, y* in $D'_a = D_a \cup \{x\} \cup E$, where

$E = \{q \in X : Tx_n, Sx_n \to q, \{x_n\} \subset D_a\}$, $0 < d < 1$, if *S* is linear, continuous on D_a and *T,S* are compatible in D_a, if D_a is nonempty, compact, convex and $S(D_a)=D_a$, then $D_a \cap F(T) \cap F(S) \neq \phi$.

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