Some Common Fixed Point Theorems For Asymptotically Commuting Mappings In Banach Spaces

A.K. Goyal, Neelmani Gupta

Abstract
Motivated and inspired by the result of Imdad, Ahamad and Khan [4] and contractive condition studied by Nesic [8], we have proved some common fixed point theorems for asymptotically commuting mappings in uniformly convex Banach spaces. Our work generalizes some known results with respect to their mappings and inequality conditions.

Keywords: Fixed points, complete metric space, Banach spaces, asymptotically commuting, weakly commuting.

2000 SUBJECT CLASSIFICATION CODE : 54H25, 47H10

1. INTRODUCTION AND PRELIMINARIES:
Let \( R_+ \) be the set of all non-negative reals and \( H_i \) be the family of all functions from \( R_+^i \) to \( R_+ \) for each positive integer \( i \), which are upper semi continuous and non decreasing in each coordinate variable.

Now, the following definitions are borrowed by several authors the weak- commutativity condition introduced by Sessa [9] in metric space, which can be described in normed linear space stated as

1.1 Definition :
Let \( A \) and \( S \) be two self mappings of a normed linear space \( X \). Then \((A, S)\) is said to be weakly commuting pair on \( X \) if
\[ \|SAx - ASx\| \leq \|Ax - Sx\| \quad \text{for all } x \in X \]
onobviously a commuting pair is weakly commuting but its converse need not be true as is evident from the following example.

1.2 Example :
Let \( X = [0,1] \) be the reals with Euclidean norm \( Ax = \frac{x}{4+x} \) and \( Sx = \frac{x}{2+x} \) for any \( x \in X \).
\[ \|SAx - ASx\| = \left| \frac{x}{8+3x} - \frac{x}{8+5x} \right| = \frac{2x^2}{(8+3x)(8+5x)} \]
\[ \leq \frac{2x}{(2+x)(4+x)} = \|Sx - Ax\| \]
So the pair \((A, S)\) is weakly commuting but it is not commuting \( SAx \neq ASx \).
The definition of compatible maps was given by Jungck [7], which can be stated as

1.3 Definition :
Let \( A \) and \( S \) be two self mappings of normed linear space \( X \). Then \((A, S)\) is said to be asymptotically or preorbitally commuting (also called compatible (Jungck [12]) its.
\[ \lim_{n \to \infty} \|SAx_n - ASx_n\| = 0 \quad \text{whenever } \{x_n\} \text{ is a sequence in } X \text{ such that} \]
\[ \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = u \text{ for some } u \in X. \]
The following example also supports the observation 1.4 Example :
Let \( X = [0, \infty) \), \( Ax = 2x^2 \), \( 5x = 3x^2 \) and \( d \) the absolute value metric on \( X \) then \( A \) and \( S \) are not weakly commuting. However, for \( x_n = 2^{-n}, d(Ax_n, Sx_n) \to 0, as \ n \to \infty \) and

1\(^*\)Department of Mathematics, M. S. J. Govt. P.G. College, Bharatpur (Raj.)-321001, Email: akgbpr67@rediffmail.com
2\(^*\)Department of Mathematics, Govt. Dungar College, Bikaner (Raj.)-334003, Email: neelmanigupta04@gmail.com

DOI: https://doi.org/10.53555/V24I10/400083
also
d(Ax_n,SAx_n)→0, as n→∞
evidently a weakly commuting pair is always asymptotically commuting but the converse is not true in general. In (1974) Iscki [5] stated as

1.5 Definition:
The modulus of convexity of a Banach space E is a function δ: (0,2]→(0,1] defined by

\[ δ(ε) = \inf \left\{ 1 - \frac{1}{2} \|x + y\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq ε \right\} \]

It is well known (Iscki [5]) that if E is uniformly convex then δ is strictly increasing. \( \lim_{ε→0} δ(ε) = 0 \) and \( δ(2) = 1 \). Let η denote the inverse of δ, then we note that \( η(t) < 2 \) for \( t < 1 \).

We shall need the following Lemma of Goebel et al. [2].

1.6 Lemma:
Let \( E \) be a uniformly convex Banach space and \( B_γ \) the closed ball in \( E \) centered at origin with radius \( γ > 0 \), if \( x_1, x_2, x_3 \in B_γ \)

\[ \|x_1 - x_2\| ≥ \|x_2 - x_3\| ≥ d > 0 \text{ and } \|x_2\| ≥ \left[ 1 - \frac{1}{2} δ\left(\frac{d}{γ}\right) \right] γ \]

then \( \|x_1 - x_3\| ≤ η\left[ 1 - \frac{1}{2} δ\left(\frac{d}{γ}\right) \right] \|x_1 - x_2\| \)

2 MAIN RESULTS
Let \( R^+ \) be the set of non-negative real numbers, and let \( F: R^+ → R^+ \) be mapping such that \( F(0) = 0 \) and \( F \) is continuous at 0.

2.1 Theorem:
Let \( E \) be a uniformly convex Banach space and \( K \), a non-empty closed subset of \( E \). Let \( \{S,I\} \) and \( \{T,J\} \) be two asymptotically commuting pairs of self-mappings of \( K \) such that for all \( x, y \in K \)

\[ ∥Sx - Ty∥^2 ≤ \phi (∥Ix - Jy∥∥Ix - Sx∥, ∥Ix - Jy∥∥Jy - Ty∥, ∥Ix - Sx∥) \]

\[ + F(\min ∥Jy - Sx∥, ∥Jy - Ty∥, ∥Ix - Sx∥, ∥Ix - Ty∥)) \]

\[ ... (1) \]

where \( φ \in H_3 \) and for all \( t > 0 \).

(i) \( φ(t,t,t,α,0) ≤ β t \), and \( φ(t,t,t,0,αt) ≤ β t \)

where \( β = 1 \) for \( α = 2 \) and \( β < 1 \) for \( α < 2 \);

(ii) \( φ(0,0,0,0,0) = 0 \);

(iii) \( I \) and \( J \) are continuous \( S(K) ⊆ J(K) \) and \( T(K) ⊆ I(K) \).

Then (a) \( S,T,I,J \) have a unique common fixed point \( z \) in \( K \) and

(b) for any \( x_0 \in K \) the sequence generated by

\[ Sx_{2n} = Jx_{2n+1}, Tx_{2n+1} = Ix_{2n+2}, n = 0,1,2,........ \]

converges strongly to \( z \).

Proof: Choose an arbitrary point \( x_0 \in K \). As \( S(K) \subseteq J(K) \), we can choose a point \( x_1 \in K \) such that \( Sx_0 = Jx_1 \). Also since \( T(K) \subseteq I(K) \), choose another point \( x_2 \in K \) such that \( Tx_1 = Ix_2 \). In this way, choose \( x_{2n}, x_{2n+1}, x_{2n+2} \) such that \( Sx_{2n} = Jx_{2n+1} \) and \( Tx_{2n+1} = Ix_{2n+2} \) for \( n = 0,1,2,.... \)
Thus we get the sequence
\[ \{ S_{x_0}, T_{x_1}, S_{x_2}, \ldots, T_{x_{2n-1}}, S_{x_{2n}}, T_{x_{2n+1}} \ldots \} \]  \hspace{1cm} \ldots (2)

Let \( d_{2n} = \| S_{x_{2n}} - T_{x_{2n+1}} \| \) and \( d_{2n+1} = \| T_{x_{2n+1}} - S_{x_{2n+2}} \| \), then

using inequality (i) we have,
\[
\| S_{x_{2n}} - T_{x_{2n+1}} \|^2 \\
\leq \phi \left( \| I_{x_{2n}} - J_{x_{2n}} \|, \| I_{x_{2n}} - S_{x_{2n}} \|, \| I_{x_{2n}} - J_{x_{2n+1}} \|, \| J_{x_{2n+1}} - T_{x_{2n+1}} \|, \right. \\
\left. \| I_{x_{2n}} - S_{x_{2n}} \|, \| J_{x_{2n+1}} - T_{x_{2n+1}} \|, \right) \\
+ F \left( \min \left\{ \| I_{x_{2n+1}} - S_{x_{2n+2}} \|, \| J_{x_{2n+1}} - T_{x_{2n+1}} \|, \| I_{x_{2n}} - S_{x_{2n}} \| \right\} \right)
\]

which implies
\[
d_{2n}^2 \leq \phi(d_{2n-1}^2, d_{2n-1}d_{2n}, d_{2n-1}d_{2n}, (d_{2n-1} + d_{2n})d_{2n-1}, 0) \\
+ F(\min\{0, d_{2n-1}^2 + d_{2n}^2\})
\]

or
\[
d_{2n}^2 \leq \phi(d_{2n-1}, d_{2n}, d_{2n}, d_{2n-1}, d_{2n}, \alpha d_{2n-1}, 0) + F(0)
\]

or
\[
d_{2n}^2 \leq \phi(d_{2n-1}, d_{2n}, d_{2n}, d_{2n-1}, d_{2n}, \alpha d_{2n-1}, 0)
\]  \hspace{1cm} \ldots (3)

Similarly, we obtain
\[
\| S_{x_{2n+2}} - T_{x_{2n+1}} \|^2 \\
\leq \phi \left( \| I_{x_{2n+2}} - J_{x_{2n+1}} \|, \| I_{x_{2n+2}} - S_{x_{2n+2}} \|, \| I_{x_{2n+2}} - J_{x_{2n+1}} \|, \| J_{x_{2n+1}} - T_{x_{2n+1}} \|, \right. \\
\left. \| I_{x_{2n+2}} - S_{x_{2n+2}} \|, \| J_{x_{2n+1}} - T_{x_{2n+1}} \|, \right) \\
+ F \left( \min \left\{ \| J_{x_{2n+1}} - S_{x_{2n+2}} \|, \| I_{x_{2n+1}} - T_{x_{2n+1}} \|, \| I_{x_{2n+2}} - S_{x_{2n+2}} \| \right\} \right)
\]

\[
d_{2n+1}^2 \leq \phi(d_{2n+1}^2, d_{2n+1}, d_{2n+1}, d_{2n+1}, 0, (d_{2n} + d_{2n+1})d_{2n+1}) \\
+ F(\min\{d_{2n+1}^2 + d_{2n+1}^2\})
\]

or
\[
d_{2n+1}^2 \leq \phi(d_{2n+1}, d_{2n+1}, d_{2n+1}, d_{2n+1}, 0, \alpha d_{2n+1}) + F(0)
\]

or
\[
d_{2n+1}^2 \leq \phi(d_{2n+1}, d_{2n+1}, d_{2n+1}, d_{2n+1}, 0, \alpha d_{2n+1})
\]  \hspace{1cm} \ldots (4)

Suppose for some \( n \) \( d_{2n+1} > d_{2n} > d_{2n-1} \), then \( d_{2n+1} + d_{2n} = \alpha d_{2n+1} \) with some \( 1 < \alpha < 2 \) and \( d_{2n} + d_{2n+1} = \alpha' d_{2n+1} \) with some \( 1 < \alpha' < 2 \) since in each coordinate, variable \( \phi \) is non decreasing.

\[
\begin{align*}
\frac{d_{2n}^2}{\alpha} & \leq \phi\left(d_{2n}^2, d_{2n}^2, \frac{d_{2n}^2}{\alpha} \right) \\
\frac{d_{2n+1}^2}{\alpha'} & \leq \phi\left(d_{2n+1}^2, d_{2n+1}^2, \frac{d_{2n+1}^2}{\alpha'} \right)
\end{align*}
\]  \hspace{1cm} \ldots (5)

In both the cases by (i), we have
\[
d_{2n}^2 \leq 0, \frac{1}{2} < \beta < 1.
\]

DOI: https://doi.org/10.53555/V24I10/400083
\[ d_{2n+1}^2 \leq \beta' d_{2n+1}^2, \quad \frac{1}{2} < \beta' < 1, \]

which is contradiction. Therefore, 
\[ d_{2n} \leq d_{2n-1} \quad \text{i.e.,} \quad d_{2n} \geq d_{2n+1}, \quad n=1,2, \ldots \]
suppose further that
\[ \lim_n \{d_{2n}\} = \lim_n \{d_{2n+1}\} = d \geq 0, \]
we claim that \( d = 0 \) and if not we can say \( d > 0 \), without loss of generality.

We can postulate that \( 0 \in K \) and \( 0 < \gamma' = \sup \|d_{2n}\| \)
Let, \( \gamma \in R_+ \) be chosen in such a way that
\[ \gamma' < \gamma \text{ and } \gamma \left[ 1-\frac{1}{2} \delta \left( \frac{d}{\gamma} \right) \right] < \gamma', \]
we can find a sequence \( \{\eta_i\}, i = 0, 1, 2, \ldots \) of positive integers such that for i.e. \( j \in \{\eta_i\} \)
\[ d_{2j} \geq \gamma \left[ 1-\frac{1}{2} \delta \left( \frac{d}{\gamma} \right) \right] \quad \text{while for} \quad \eta > \eta_0, \quad d_{2\eta} \leq \gamma \]
since \( d_{2\eta_1} \geq d_{2\eta_j} \geq d \geq 0 \) for every \( i = 0, 1, 2, \ldots \)

It follows from Lemma 1.6, it follows that for any \( j \in \{\eta_i\} \)
\[ \|S_{x_{2j-2}} - S_{x_{2j}}\| \leq \|S_{x_{2j-2}} - T_{x_{2j-1}}\| + \|T_{x_{2j-1}} - S_{x_{2j}}\| \]
\[ \leq \eta \left[ 1-\frac{1}{2} \delta \left( \frac{d}{\gamma} \right) \right] \|S_{x_{2j-2}} - T_{x_{2j-1}}\| \]
\[ \leq \eta \left( \frac{\gamma'}{\gamma} \right) d_{2j-1} = \alpha_j d_{2j-1} - 1 \quad \ldots (6) \]
where \( \alpha_j = \eta \left( \frac{\gamma'}{\gamma} \right) < 2 \) because of uniform convexity.

Then we have by (3), (4), (5) and (6)
\[ \left\{ \begin{array}{l}
\frac{\delta^2 d_{2j-1}^2}{\phi(d_{2j-1}^2, d_{2j-1}^2 d_{2j-1}^2, 0, d_{2j-1}^2)} \\
\frac{\delta^2 d_{2j-1}^2}{\phi(d_{2j-1}^2, d_{2j-1}^2 d_{2j-1}^2, 0, d_{2j-1}^2)}
\end{array} \right\} \quad \ldots (7) \]

Thus in either case \( d_{2j} \leq \beta_1 d_{2j-1} \) and \( d_{2j+1} \leq \beta_1 d_{2j} \) for some \( \beta_1 < 1 \).

We observed that \( \beta_1 \) is independent of \( j \) and so, as \( j \to \infty \), we have \( d \leq \beta_1 d \), a evident contradiction implying at \( d = 0 \).

It follows therefore, as proved in (Husain and Sehgal [3] that the sequence (2) is cauchy sequence. But \( K \) is closed subset of \( E \), therefore sequence (2) converges to a point \( z \) in \( K \), hence the sequence \( \{S_{x_{2n}}\} = \{I_{x_{2n+1}}\} \) and \( \{T_{x_{2n-1}}\} = \{I_{x_{2n}}\} \) which are subsequences of (1) also converges to the point \( z \).

Since \( I \) is continuous then sequence \( Fx_{2n} \) and \( IS_{x_{2n}} \) converges to \( Iz \). since \( \lim_{n \to \infty} S_{x_{2n}} = \lim_{n \to \infty} I_{x_{2n}} = z \) and \( (S,I) \) is asymptotically commuting, then
\[ \lim_{n \to \infty} \|SI_{x_{2n}} - IS_{x_{2n}}\| = 0 \]
which implies that \( SI_{x_{2n}} \to Iz \)
Taking \( x = I_{x_{2n}}, \ y = x_{2n+1} \), in condition (1)
\[ \|Sx_{2n} - Tx_{2n+1}\|^2 \leq \phi \left( \|I^2x_{2n} - J^2x_{2n+1}\| \right) \left( \|I^2x_{2n} - Jx_{2n+1}\| \right) \left( \|J^2x_{2n+1} - Jx_{2n+1}\| \right) \left( \|J^2x_{2n} - Sx_{2n}\| \right) \left( \|J^2x_{2n} - Jx_{2n+1}\| \right) \left( \|J^2x_{2n+1} - TJx_{2n+1}\| \right) \]

Taking \( \lim n \to \infty \), we have
\[
\|Iz - z\|^2 \leq \phi \left( \|Iz - z\| \right) \left( \|Iz - Iz\| \right) \left( \|z - z\| \right) \left( \|z - Iz\| \right) \left( \|z - z\| \right) + F \left( \min \{ \|z - Iz\|, \|z - z\|, \|z - Iz\| \} \right)
\]

or \( \|Iz - z\|^2 \leq \phi (0,0,0,0,0) + F \left( \min \{ \|z - Iz\|, 0, 0, \|z - z\| \} \right) \)

or \( \|Iz - z\|^2 \leq 0 + F(0) \)

which implies that \( Iz = z \).

Since \( J \) in continuous and \((T,J)\) in asymptotically commuting. So the sequence \( J^2x_{2n+1} \to Jz \), \( JTx_{2n+1} \to Jz \).

Since \( \lim_{n \to \infty} Tx_{2n+1} = \lim_{n \to \infty} Jx_{2n+1} = z \) while \((T,J)\) is asymptotically commuting then

\( \lim_{n \to \infty} \|TJx_{2n+1} - JTx_{2n+1}\| = 0 \),

which implies that \( TJx_{2n+1} \to Jz \).

Now, putting \( x = x_{2n} \), \( y = Jx_{2n+1} \), in condition (1), we have
\[
\|Sx_{2n} - TJx_{2n+1}\|^2 \leq \phi \left( \|Iz - z\| \right) \left( \|Iz - Iz\| \right) \left( \|z - z\| \right) \left( \|z - Iz\| \right) + F \left( \min \{ \|z - Iz\|, \|z - z\|, \|z - Iz\| \} \right)
\]

or \( \|Iz - z\|^2 \leq \phi (0,0,0,0,0) + F \left( \min \{ \|z - Iz\|, 0, 0, \|z - z\| \} \right) \)

or \( \|z - Jz\|^2 \leq 0 + F(0) \)

giving there by \( z = Jz \) which implies \( z = Jz = Iz \)

Taking \( x = z, y = x_{2n+1} \), in condition (1), we have
\[
\|Sz - Tx_{2n+1}\|^2 \leq \phi \left( \|Iz - z\| \right) \left( \|Iz - Iz\| \right) \left( \|z - z\| \right) \left( \|z - Iz\| \right) + F \left( \min \{ \|z - Iz\|, \|z - z\|, \|z - Iz\| \} \right)\]

or \( \|z - Jz\|^2 \leq \phi (0,0,0,0,0) + F \left( \min \{ \|z - Iz\|, 0, 0, \|z - z\| \} \right) \)

or \( \|z - Jz\|^2 \leq 0 + F(0) \)
Taking $\lim n \to \infty$, we have
\[ \|S_{x_n} - T_z\|^2 \leq \phi(0,0,0,0) + F\left(\min\{\|z - S_z\|, \|z - T_z\|, \|I_z - S_z\|, \|I_z - T_z\|\}\right) \]
\[ \text{or} \quad \|S_{x_n} - T_z\|^2 \leq 0 + F(0) \]
yielding thereby $S_z = z$.

Now, taking $x = x_{2n}$, $y = y$, in condition (1), we have
\[
\begin{aligned}
&\|S_{x_n} - T_z\|^2 \\
\leq & \phi(\|I_{x_{2n}} - J_z\|, \|I_{x_{2n}} - S_{x_{2n}}\|, \|I_{x_{2n}} - J_z\|, \|J_z - T_z\|, \|I_{x_{2n}} - S_{x_{2n}}\|) \\
&+ F\left(\min\{\|J_z - S_{x_{2n}}\|, \|J_z - T_z\|, \|I_{x_{2n}} - S_{x_{2n}}\|, \|I_{x_{2n}} - T_z\|\}\right) \\
\end{aligned}
\]
Taking $\lim n \to \infty$, we have
\[ \|z - T_z\|^2 \leq \phi(0,0,0,0,0) + F\left(\min\{0, \|z - T_z\|, 0, \|z - T_z\|\}\right) \]
\[ \text{or} \quad \|z - T_z\|^2 \leq 0 + F(0) \]
yielding thereby $z = T_z$ which implies $z = S_z = T_z$. Thus we have proved that $z = S_z = I_z = T_z = J_z$, So $z$ is the common fixed point of $S, I, T$ and $J$. This completes the proof.

If we take $F(t) = 0$ and for all $t \in R^+$, in Theorem 2.1, we obtain the following result.

2.2 Corollary:
Let $E$ be uniformly convex Banach space and $K$ a non empty closed subset of $E$. Let $\{S, I\}$ and $\{T, J\}$ be two asymptotically commuting pairs of self mappings of $K$ such that for all $x, y \in K$,
\[ \|S_{x_{2n}} - T_{z_n}\|^2 \leq \phi(\|I_{x_{2n}} - J_z\|, \|I_{x_{2n}} - S_{x_{2n}}\|, \|I_{x_{2n}} - J_z\|, \|J_z - T_z\|, \|I_{x_{2n}} - S_{x_{2n}}\|) \\
\[
\text{or} \quad \|z - T_z\|^2 \leq \phi(0,0,0,0,0) + F\left(\min\{0, \|z - T_z\|, 0, \|z - T_z\|\}\right) \\
\text{or} \quad \|z - T_z\|^2 \leq 0 + F(0)
\]
where $\phi \in H_3$ and for all $t > 0$,
(i) $\phi(t, t, t, 0, t) \leq \beta t$ and $\phi(t, t, t, 0, t) \leq \beta t$ where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$;
(ii) $\phi(0,0,0,0,0) = 0$;
(iii) $I$ and $J$ are continuous $S(K) \subset J(K)$ and $T(K) \subset I(K)$

Then (a) $S, T, I$ and $J$ have a unique common fixed point $z$ in $K$ and.
(b) for any $x_0 \in K$ the sequence generated by
\[ S_{x_{2n}} = J_{x_{2n}+1}, T_{x_{2n}+1} = I_{x_{2n}+2}, n = 0,1,2,... \]
converges strongly to $z$.

The above corollary (2.2) shows the result of Imdad, Ahmad and Khan [4].

The following theorem also generalizes the result of Som [10] and Imdad, Ahmad and Khan [4].

2.3 Theorem:
Let $E$ be uniformly convex Banach space and $K$ a non empty closed subset of $E$. Let $S, T, I$ and $J$ be four self mapping of $K$ such that for all $x, y \in K$.
\[ \|Sx - Ty\| \leq \phi \left( \|Ix - Sx\|, \|Jy - Ty\|, \|Ix - Ty\|, \|Ix - Sx\|, \|y - Ty\| \right) \]

where \( \phi \in H_4 \) and for all \( t > 0 \),

(i) \( \phi(t,0,0,0,0) \leq \beta t \) and \( \phi(t,0,0,0,0) \leq \beta t \) where \( \beta = 1 \) for \( \alpha = 2 \) and \( \beta < 1 \) for \( \alpha < 2 \);

(ii) \( \phi(t,0,0,0) < t, \phi(0,0,0,0) = 0 \);

(iii) \( S(K) \subset J(K) \) and \( T(K) \subset I(K) \);

(iv) \( I \) is continuous, \( (S,I) \) is asymptotically commuting and \( (T,J) \) is weakly commuting pair in \( K \) or \( J \) is continuous, \( (T,J) \) is asymptotically commuting and \( (S,I) \) is weakly commuting pair in \( K \).

Then

(a) \( S, T, I \) and \( J \) have a unique common fixed point \( z \) in \( K \) and.

(b) for any \( x_0 \in K \) the sequence generated by

\[ Sx_{2n} = Jx_{2n+1}, Tx_{2n+1} = Ix_{2n+2}, \quad n = 0,1,2,... \]

converges strongly to \( z \).

\textbf{Proof :} It may be completed on the lines of proof of Theorem 2.1

If we take \( I = J \) and \( F(t) = 0 \), for all \( t \in \mathbb{R}^+ \) in Theorem 2.3, we get the following result of Som [10].

\textbf{2.4 Corollary :}

Let \( E \) be a uniformly convex Banach space and \( K \) a non-empty closed subset of \( E \). Let \( S, T \) and \( I \) be three self mappings of \( K \) such that for all \( x, y \in K \).

\[ \|Sx - Ty\| \leq \phi \left( \|Ix - Sx\|, \|Jy - Ty\|, \|Ix - Ty\|, \|Ix - Sx\|, \|y - Ty\| \right) \]

where \( \phi \in H_4 \) and for all \( t > 0 \),

(i) \( \phi(t,0,0,0) \leq \beta t \) and \( \phi(t,0,0,0) \leq \beta t \) where \( \beta = 1 \) for \( \alpha = 2 \) and \( \beta < 1 \) for \( \alpha < 2 \);

(ii) \( \phi(t,0,0,0) < t, \phi(0,0,0,0) = 0 \);

(iii) \( S(K) \cup T(K) \subset I(K) \);

(iv) \( (S,I) \) and \( (T,J) \) are asymptotically commuting pairs on \( K \). Then there exists a point \( u \in K \) such that

(a) \( u \) is the unique common fixed point of \( S, T, I \) and \( J \);

(b) for any \( x_0 \in K \), the sequence \( \{Ix_n\} \) defined by

\[ Ix_{2n} = Sx_{2n+1}, Tx_{2n+1} = Jx_{2n+2}, \quad n = 0,1,2,... \]

converges strongly to \( u \).

Finally, we furnish the example to discuss the validity of foregoing Theorem 2.1

\textbf{2.5 Example :} Let \( E = K \) \([0, 1]\) and define

\[ S, T, I, J : K \to K \text{ as } Sx = \frac{x^2}{3}, Tx = \frac{x^2}{4}, Jx = \frac{x}{2} \text{ and } Ix = \frac{3x}{4} \]

\textbf{Note that} \( S(K) \subset J(K) \)

\[ \Rightarrow S(K) = \left[ 0, \frac{1}{3} \right] \subset \left[ 0, \frac{1}{2} \right] = J(K) \]

Let \( \phi(t_1, t_2, t_3, t_4, t_5) = \frac{1}{5} (t_1 + t_2 + t_3 + t_4 + t_5) \)

Then above function satisfy condition (4.1) for all \( x, y \text{ in } [0,1] \). Clearly 0 is the unique fixed point of \( S, T, I \) and \( J \). taking \( F(t) = 0 \) for all \( t \in \mathbb{R}^+ \)

Next example proof the validity of corollary (2.4)
2.6 Example:

Let $K = [0,1]$. Define the mappings $S$, $T$ and $I$ of $K$ into itself by $Sx = \frac{x^2}{4}, Tx = \frac{x^2}{3}$ and $Ix = \frac{3x}{4}$.

Let $\phi(t_1,t_2,t_3,t_4) = \frac{6}{25}(t_1 + t_2 + t_3 + t_4)$ then all the conditions of corallary (2.4) are satisfied and 0 is the unique common fixed point of $S$, $T$ and $I$.

Next example proof the validity of Theorem 2.3

2.7 Example:

Let $K = [0,1]$. Define the mappings $S$, $T$, $I$ and $J$ of $K$ into itself by $Sx = \frac{x^2}{3}, Tx = \frac{x^2}{4}$, $Ix = \frac{3x}{4}$ and $Jx = \frac{x}{2}$

If we set $\phi(t_1,t_2,t_3,t_4) = \frac{1}{4}(t_1 + t_2 + t_3 + t_4)$ then all the conditions of Theorem 2.3 are satisfied for all $x, y \in [0,1]$. Clearly 0 is the unique common fixed point of $S$, $T$, $I$ and $J$.

REFERENCES