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# Equivalence Relations and Bell Numbers

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***Abstract***

*It is well known that for a given finite set, an equivalence relation induces a partition of the set. This paper addresses the question of counting the number of equivalence relations that can be defined on a given finite set. Interestingly enough the answer lies in special class of numbers called "Bell Numbers". In this paper, we witness this amusing connection obtained through another special class of numbers called Stirling's numbers of second kind. Some of the basic properties of Stirling's numbers and Bell numbers were proved.*

***Keywords****: Partitions, Equivalence relations, Stirling's numbers of second kind, Recurrence relation, Bell numbers, Exponential generating function.*

## I. Introduction

In the study of branch of mathematics called "Discrete Mathematics", we often consider the idea of equivalence relations defined on a given set. In this paper, we try to enumerate the number of such equivalence relations defined on a finite set. Using the interesting class of numbers like Stirling's numbers of second kind (there is another class called Stirling's numbers of first kind) we would obtain the famous Bell numbers. While doing this, we prove that the total number of equivalence relations that can be defined on a given finite set is precisely the Bell numbers. The Stirling numbers of both kinds are named after Scottish mathematician James Stirling and Bell numbers are named after another Scottish mathematician Eric Temple Bell, who is the author of the most celebrated book "Men of Mathematics".

## II. Definitions

We will begin our paper with the followingdefinitions.

**2.1 Partition of a Set**

A Partition of a set *S* is a collection of non-empty subsets of *S* of the form $A_i$, $i = 1, 2, 3, \ldots, m$ such that $\bigcup_{i=1}^{m} A_i = S$ and $A_i \cap A_j = \phi$ whenever $i \neq j$. Thus, the partition of a set splits the given set in to disjoint subsets whose union is the given set *S*. The sets $A_i$ are called parts of the partition or components of the partition.

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For example, $\{\{1\},\{2,3\}\}$ is a partitionof the set of the set $S=\{1,2,3\}$. Similarly, the set of odd integers and even integers form a partition of set of all integers.

**2.2 Equivalence Relation**

A relation defined between two sets is called as Equivalence Relation if it is Reflexive, Symmetric and Transitive.

For example, the relation *R* defined on the set $S=\{1,2,3,4,5,6,7\}$ by $xRy$ if and only if $x-y$ is divisible by 3, being reflexive, symmetric and transitive is an equivalence relation on *S*. We notice that this equivalence relation will give rise to the partition $\{\{1,4,7\},\{2,5\},\{3,6\}\}$ of the set *S*. In view of Definition 2.1, we see that the parts of the partition are given by the sets $A_1=\{1,4,7\}$, $A_2=\{2,5\}$, $A_3=\{3,6\}$. It is clear that $A_1$, $A_2$, $A_3$ are disjoint subsets of *S* whose union is the whole set *S*.

**2.3 Stirling's Numbers of Second Kind**

Let *S* be a set with *n* elements. We define the Stirling's numbers of second kind denoted by $S(n,k)$ or $\left\{\begin{matrix} n \\ k \end{matrix}\right\}$ as the number of partitions of *S* containing exactly *k*parts. That is the Stirling's numbers of second kind represent the number of partitions of a set with *n* elements using *k* non-empty disjoint subsets.

In this sense, it follows that $0 \le k \le n$. In particular, if $n = 0$, $k = 0$ we consider $S(0,0)=\left\{\begin{matrix} 0 \\ 0 \end{matrix}\right\}=1$. Similarly there is only one possible partition namely the whole set *S* itself if $k = n$. Thus, $S(n,n)=\left\{\begin{matrix} n \\ n \end{matrix}\right\}=1$. Also, if $k>n$ then there is no possibility of obtaining any partition of *S* with more than *n* non-empty subsets (since the minimum cardinality must be 1). Hence, $S(n,k)=\left\{\begin{matrix} n \\ k \end{matrix}\right\}=0$ if $k>n$.

**2.3.1** If $S=\{a,b,c,d\}$ then $n = 4$. If we now try to count the number of partitions of *S* in to 1,2,3,4 non-empty disjoint subsets then we have:

With 1 subset we have only **1** partition namely*S* itself given by $\{\{a,b,c,d\}\}$.

With 2 subsets we have **7** possible partitions given by

$$\{\{a\},\{b,c,d\}\};\{\{b\},\{a,c,d\}\};\{\{c\},\{a,b,d\}\};\{\{d\},\{a,b,c\}\};\{\{a,b\},\{c,d\}\};\{\{a,c\},\{b,d\}\};$$
$$\{\{a,d\},\{b,c\}\}$$

With 3 subsets we have 6 possible partitions given by

$$\{\{a\},\{b\},\{c,d\}\};\{\{a\},\{c\},\{b,d\}\};\{\{a\},\{d\},\{c,d\}\};\{\{b\},\{c\},\{a,d\}\};$$
$$\{\{b\},\{d\},\{a,c\}\};\{\{c\},\{d\},\{a,b\}\}$$

With 4 subsets we have just 1 possible partition given by $\{\{a\},\{b\},\{c\},\{d\}\}$. In this case, we notice that all the parts of the partition are singleton sets.

Thus, as in definition 2.3, we have

$$S(4,1)=\left\{\begin{matrix}4\\1\end{matrix}\right\}=1, S(4,2)=\left\{\begin{matrix}4\\2\end{matrix}\right\}=7, S(4,3)=\left\{\begin{matrix}4\\3\end{matrix}\right\}=6, S(4,4)=\left\{\begin{matrix}4\\4\end{matrix}\right\}=1.$$

## III. Stirling’sNumbers of Second Kind Triangle

With the aid of definitions and examples presented in section 2, we can construct a triangle portraying Stirling’s numbers of second kind as shown in Figure 1.

| n \ k | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | |
| 1 | 0 | 1 | | | | | | | | | |
| 2 | 0 | 1 | 1 | | | | | | | | |
| 3 | 0 | 1 | 3 | 1 | | | | | | | |
| 4 | 0 | 1 | 7 | 6 | 1 | | | | | | |
| 5 | 0 | 1 | 15 | 25 | 10 | 1 | | | | | |
| 6 | 0 | 1 | 31 | 90 | 65 | 15 | 1 | | | | |
| 7 | 0 | 1 | 63 | 301 | 350 | 140 | 21 | 1 | | | |
| 8 | 0 | 1 | 127 | 966 | 1701 | 1050 | 266 | 28 | 1 | | |
| 9 | 0 | 1 | 255 | 3025 | 7770 | 6951 | 2646 | 462 | 36 | 1 | |
| 10 | 0 | 1 | 511 | 9330 | 34105 | 42525 | 22827 | 5880 | 750 | 45 | 1 |

**Figure 1: Stirling Numbers of Second Kind Triangle**

### 3.1 BellNumbers

The sum of each row numbers in Figure 1 containing Stirling numbers of second kind are called Bell's Numbers. If we do so, then from Figure 1, we get 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, . . . We denote the *n*th Bell number by $B_n$. Thus $B_0 = 1, B_1 = 1, B_2 = 2, B_3 = 15, B_4 = 52, B_5 = 203, B_6 = 877, ...$

The sequence of first twenty Bell numbers are given by 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, 27644437, 190899322, 1382958545, 10480142147, 82864869804, 682076806159, 5832742205057, . . .

We now present the following important theorem.

### 3.2 Theorem 1

The total number of equivalence relations that can be defined on a set with *n* elements is the *n*th Bell number $B_n$.

**Proof**: By definition of Stirling numbers of second kind, we know that $S(n,k)$ represent the number of partitions of a set with *n* elements using *k* non-empty disjoint subsets where $0 \le k \le n$. Thus the total number of possible partitions that can be obtained for a set with *n* elements will be sum of all Stirling's numbers of second kind for each value of *k* given by $\sum_{k=0}^{n} S(n,k) \quad (3.1)$. At the same time, $\sum_{k=0}^{n} S(n,k)$ also represents the sum of all numbers in row *n* of the Stirling numbers of second kind triangle of Figure 1. But by definition of Bell numbers, this sum is precisely the *n*th Bell number by $B_n$. Hence, $B_n = \sum_{k=0}^{n} S(n,k) = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \quad (3.2)$.

We know that any equivalence relation defined on a set produces a partition of that set. Hence, the total number of equivalence relations defined on a set with *n* elements must be same as that of the number of partitions that can be obtained for that set. But the total number of partitions for a set with *n* elements from (3.2) is the *n*th Bell number by $B_n$. Thus, the total number of equivalence relations on a set with *n* elements is precisely $B_n$. This completes the proof.

We notice that while proving this theorem, we have obtained a fact that the total number of partitions of a set with *n* elements is $B_n$ given by equation (3.2). We first, note that the binomial coefficients representing number of *k* elements subsets from a set of *n* elements is given by $\binom{n}{k} = \frac{n!}{k! \times (n-k)!} \quad (3.3)$. We now proceed to prove an interesting recurrence relation concerning Bell numbers through the following theorem.

**3.3 Theorem 2**

If $B_k$ is the $k$th Bell number, then $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$ (3.4)

**Proof**: Let $A_1, A_2, A_3, ..., A_m$ be parts of a partition of the set $S = \{1, 2, 3, ..., n, n+1\}$. Without loss of generality, we may assume that $n + 1$ is in $A_1$and that $A_1$contain $k + 1$ elements of $S$, where $0 \le k \le n$. Then, $A_2, A_3, ..., A_m$ forms a partition of the remaining $n-k$ elements of $S - A_1$. By theorem 1, we know that the total number of partitions of $S - A_1$ containing $n-k$ elements must be $B_{n-k}$. Hence there are $B_{n-k}$ partitions of $S$ in which one part is $A_1$. Now we notice that there will be $\binom{n}{k}$ sets of size $k + 1$ containing $n + 1$. Hence the total number of partitions of $S$ in which $n + 1$is in a set of size $k + 1$ is $\binom{n}{k} B_{n-k}$. If we add these values for all possible values of $k$ from 0 to $n$, that gives the total possible partitions of the set $S$, which must be $B_{n+1}$. Hence, we have $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}$

Now this equation can be re-written as $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} = \sum_{k=0}^{n} \binom{n}{n-k} B_{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_k$

This completes the proof.

Equation (3.4) enables us for generating successive Bell numbers. The following calculations provide the first five Bell numbers using the formula derived in Theorem 1.

$$B_1 = \sum_{k=0}^{0} \binom{0}{k} B_0 = 1 \times 1 = 1$$

$$B_2 = \sum_{k=0}^{1} \binom{1}{k} B_k = \binom{1}{0} B_0 + \binom{1}{1} B_1 = (1 \times 1) + (1 \times 1) = 2$$

$$B_3 = \sum_{k=0}^{2} \binom{2}{k} B_k = \binom{2}{0} B_0 + \binom{2}{1} B_1 + \binom{2}{2} B_2 = (1 \times 1) + (2 \times 1) + (1 \times 2) = 5$$

$$B_4 = \sum_{k=0}^{3} \binom{3}{k} B_k = \binom{3}{0} B_0 + \binom{3}{1} B_1 + \binom{3}{2} B_2 + \binom{3}{3} B_3 = (1 \times 1) + (3 \times 1) + (3 \times 2) + (1 \times 5) = 15$$

$$B_5 = (1 \times 1) + (4 \times 1) + (6 \times 2) + (4 \times 5) + (1 \times 15) = 52$$

We notice that though equation (3.4) helps us to generate Bell numbers successively, it is not very efficient especially when $n$ is large. We know that the entries of the Pascal triangle are the binomial coefficients $\binom{n}{k}$. Hence, by theorem 1, we see that by considering the entries of Pascal triangle in each row and multiplying them by the known Bell numbers successively with the corresponding binomial coefficient and adding up all the numbers, we can immediately get the next Bell number.

## IV. Bell Triangle

We will now consider the construction rule for a triangle known as Bell Triangle. This triangle provides the way of generating Bell numbers quite easily compared to that of the result arrived in theorem 1. To construct the Bell triangle, we consider a recurrence relation similar to that of in Pascal's Triangle.

The entries of the Bell triangle are defined recursively through the relation

$b_{n,k} = b_{n,k-1} + b_{n-1,k-1}$ (5.1) where $b_{1,1} = 1$ and $1 \le k \le n$.

Using (5.1), we get a triangle for the first seven rows as shown in Figure 2.

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 1 | 2 | | | | | |
| 2 | 3 | 5 | | | | |
| 5 | 7 | 10 | 15 | | | |
| 15 | 20 | 27 | 37 | 52 | | |
| 52 | 67 | 87 | 114 | 151 | 203 | |
| 203 | 255 | 322 | 409 | 523 | 674 | 877 |

**Figure 2: Bell Triangle**

The Bell Triangle displayed above is constructed in a such a way that the first row entry is 1 since $b_{1,1}$= 1. The other entries in the triangle is constructed in such a way that any number in a particular row and column is sum of two numbers located just left to it and vertically above to this entry. For example, from Figure 2, we notice that 7 + 20 = 27, 37 + 114 = 151, 67 + 255 = 322. Further, the first number in any row is the last number in the previous row. This implies that the first column entries are equal to the leading diagonal entries which are precisely the Bell numbers justifying the name assigned to it. This is perhaps the easiest possible way to generate Bell numbers.

# V. Properties of Stirling and Bell Numbers

In this section, we prove some of the interesting properties concerningStirlingnumbers of second kind and Bell numbers.

**5.1 Theorem 3**

The Stirling numbers of second kind satisfy the following equations

(a) $S(n,2)=\left\{\begin{matrix} n \\ 2 \end{matrix}\right\}=2^{n-1}-1 \quad (6.1)$

(b) $S(n,n-1)=\left\{\begin{matrix} n \\ n-1 \end{matrix}\right\}=\binom{n}{2}=\frac{n(n-1)}{2} \quad (6.2)$

**Proof**:

(a) If we want to make partition of a set containing $n$ elements in to 2 parts of partition then such a partition would be of the form: first part with any one of the $n$elements and second with remaining $n$ – 1 elements (or) first part with any two of the elements and second with remaining $n$ – 2 elements and so on up to first part with $n$ – 1 elements and second with just 1 element. But this list represents the ordered sets of partitions which is twice the unordered sets of partitions. Hence, the total number of unordered sets of partitions would be $\frac{1}{2}\left[\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-1}\right]$.

But we know that $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-1}+\binom{n}{n}=2^n$.

Thus, $\frac{1}{2}\left[\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-1}\right]=\frac{1}{2}\left[2^n-2\right]=2^{n-1}-1$. Thus, $S(n,2)=\left\{\begin{matrix} n \\ 2 \end{matrix}\right\}=2^{n-1}-1$.

(b) If we want to make partition of a set with $n$ elements in to $n$ – 1 parts, then in such a partition one part will have two elements and the remaining $n$ – 2 parts will each be containing exactly one element. Thus to obtain such a partition is to choose any two elements out of $n$ possible elements which is clearly $\binom{n}{2}=\frac{n(n-1)}{2}$. Thus $S(n,n-1)=\left\{\begin{matrix} n \\ n-1 \end{matrix}\right\}=\binom{n}{2}=\frac{n(n-1)}{2}$. This completes the proof.

We can view the result (a) as entries of third column for $k$ = 2 in Figure 1. Similarly we can see the result (b) along second leading diagonal giving triangular numbers in Figure 1. These values serve as verification of results obtained in theorem 3.

We now prove a theorem regarding Bell numbers.

**5.2 Theorem 4**

The *n*th Bell number $B_n$ is given by $B_n = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}$ (6.3)

**Proof**: One of the methods of generating the *n*th Bell number $B_n$ is by observing it as the coefficient of $\frac{x^n}{n!}$ in the Mclaurin's series expansion of the exponential generating function $e^{e^x-1}$. With this convention, we have

$$e^{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (6.4)$$

Now by definition of exponential series, we have $e^{e^x} = \sum_{k=0}^{\infty}\frac{e^{kx}}{k!} = \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{n=0}^{\infty}\frac{(kx)^n}{n!}$.

Hence, $e^{e^x-1} = \frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{n=0}^{\infty}\frac{(kx)^n}{n!}$ (6.5). From equations (6.4) and (6.5) we have

$$\sum_{k=0}^{\infty} B_n \frac{x^n}{n!} = \frac{1}{e}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{n=0}^{\infty}\frac{(kx)^n}{n!}.$$

Now equating coefficients of $\frac{x^n}{n!}$ on both sides we get $B_n = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}$ which is equation (6.3). This completes the proof.

## VI. Conclusion

We have discussed the concepts of Partitions that can be made in a given finite set and proved a significant property in theorem 1, that the total number of possible equivalence relations that can be defined on a set with *n* elements is precisely the *n*th Bell number $B_n$. The Stirling's numbers of second kind is introduced and through them we have constructed Bell numbers. Two triangles portraying these numbers were displayed in Figures 1 and 2 respectively. In theorem 2, we proved that the $(n+1)$th Bell number can be obtained through the first *n* Bell numbers. Though this formula helps us to generate successive Bell numbers, this is not efficient. To overcome this problem, we have provided a nice scheme forming Bell triangle which contain all Bell numbers. Through theorem 3, we have proved two interesting and important properties of Stirling's numbers of second kind and finally in theorem 4, we proved a formula for the *n*th Bell number using exponential generating function.

This paper covers some of the significant aspects of Bell numbers. There are few more formulas that can be proved regarding Stirling's numbers of second kind. Moreover just like Fibonacci and Catalan numbers, Bell numbers also have abundant connection with so many counting problems many of which are considered classic results in Combinatorics. Interested readers may try to know them and discover something new on their own.

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