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Solution of nonlinear Volterra integro-differential equations of the second kind using Accelerated Adomian Polynomials

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Abstract

In this paper, an accelerated Adomian polynomials deduced by EL-Kalla (2005, 2007) are used to solve a class of nonlinear integro-differential equations. Some examples investigated and it has found that EL-Kalla formula converges rapidly than the traditional formula of Adomian and the Laplace-Adomian modification, moreover it is programmable, and it saves time to get the approximate solutions with the same version of Matlab. Also, error analysis is established for every example.

Keywords: *Adomian decomposition method, Adomian polynomials, El-kalla polynomials and nonlinear integro-differential equations, Laplace-Adomian modification.*

I. Introduction

The Adomian Decomposition Method (ADM) solves successfully different types of linear and nonlinear equations in deterministic or stochastic fields [1–4]. Application of ADM to different types of integro-differential equations has been discussed by many authors, for example [5–8].

In this work, the nonlinear Volterra integro-differential equation of the second kind we consider

$$u^{(n)}(x) = f(x) + \int_{0}^{x} k(x,t) f\big(u(t)\big) dt \qquad ,(1)$$

where $u^{(n)}\ (x) = \frac{d^n u}{dx^n} = L$, $k(x,t)$ is the kernel of the integro-differential equation and $f\big(u(t)\big)$ is the term of nonlinearity.

II. The Method

we know that the Adomian polynomial representation is

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$$f(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots\dots.., u_n) \quad , \qquad (2)$$

where the traditional formula of Adomian's polynomials for A_n is

$$A_n = \left(\frac{1}{n!}\right)\left(\frac{d^n}{d\lambda^n}\right)\left[f\left(\sum_{n=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0} . \qquad (3)$$

2.1 Accelerated formula

The author in [9] deduced another accelerated and programmable formula for the Adomian polynomials (El-Kalla polynomials), such that

$$\overline{A_n} = f(S_n) - \sum_{i=0}^{n-1} \bar{A}_i \; , \qquad (4)$$

where the partial sum is

$$S_n = \sum_{i=0}^{n} u_i\,(x) \; .$$

Application of ADM to (1) yields

$$u(x) = \sum_{i=0}^{n} u_i\,(x), \qquad (5)$$

where

$$u_0(x) = \emptyset(x) + L^{-1}\, f(x)dx \qquad (6)$$

$$u_i(x) = L^{-1} \int_0^x k(x,t)\, \bar{A}_{i-1}\, dt, \quad i \geq 1 \quad . \qquad (7)$$

2.2 The combined Laplace-Adomian method

The author in [10] applying the Laplace transform to both sides of (1) such that

$$s^n \mathcal{L}\{u^n(x)\} - s^{n-1}u(0) - s^{n-2}u'(0) - \cdots - u^{n-1}(0)$$
$$= \mathcal{L}\{f(x)\} + \mathcal{L}\{k(x-t)\}\mathcal{L}\{f(u(t))\}. \qquad (8)$$

This can be reduced to

$$\mathcal{L}\{u(x)\} = \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \cdots + \frac{1}{s^n}u^{n-1}(0)$$
$$+ \frac{1}{s^n}\mathcal{L}\{f(x)\} + \frac{1}{s^n}\mathcal{L}\{k(x-t)\}\mathcal{L}\{f(u))\}. \qquad (9)$$

And the solution will be

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} u_n(x)\right\} = \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \cdots + \frac{1}{s^n}u^{n-1}(0) + \frac{1}{s^n}\mathcal{L}\{f(x)\}$$
$$+ \frac{1}{s^n}\mathcal{L}\{k(x-t)\}\mathcal{L}\left\{\sum_{n=0}^{\infty} A_n(x)\right\}. \quad (10)$$

Where

$$\mathcal{L}\{u_0(x)\} = \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) + \cdots + \frac{1}{s^n}u^{n-1}(0) + \frac{1}{s^n}\mathcal{L}\{f(x)\}, \qquad (11)$$

$$\mathcal{L}\{u_{n+1}(x)\} = \frac{1}{s^n}\mathcal{L}\{k(x-t)\}\mathcal{L}\left\{\sum_{n=0}^{\infty} A_n(x)\right\}, n \geq 0. \quad (12)$$

III. Convergence Remarks

Convergence of the Adomian method when applied to some classes of integro- differential equations is discussed by many authors. For example, El-Kalla [9] proved the convergence of the Adomian method for a class of Volterra type integro-differential equations. Also, Wazwaz and Khuri discussed applications of the Adomian decomposition method to a class of Fredholm integral equations that occurs in acoustics [10]. Zhang [11] presented a modified Adomian decomposition method to solve a class of nonlinear singular boundary-value problems, which arise as normal model equations in nonlinear conservative systems. Zhu et al. [12] presented a new algorithm for calculating Adomian polynomials for nonlinear operators. Also, many modifications were made to this method by numerous researchers in an attempt to improve the accuracy or extend the applications of this method [13]. Also, El-Kalla polynomial was discussed by El-Kalla in [14, 15, 16, 17, 18], and conclude that El-kalla polynomial was directly used to estimate the maximum absolute truncated error of the Adomian series solution which cannot be estimated using the traditional polynomials.

IV. Numerical experiments

The Adomian polynomials can be generated using formula (3) or **(**El-Kalla formula**)** formula (4). Formula (4) is programmable and the Adomian series solution can be converged faster when using it. For example, if $f(u) = u^2$, the first four polynomials using formulas (3) and (4) are computed to be:

Using formula (3):

$$A_0 = u_0^2$$

$$A_1 = 2u_0u_1$$

$$A_2 = u_1^2 + 2u_0u_2$$

$$A_3 = 2u_1u_2 + 2u_0u_3$$

$$A_4 = u_2^2 + 2u_1u_3 + 2u_0u_4$$

Using (El-Kalla formula) formula (4):

$$\overline{A_0} = u_0^2$$

$$\overline{A_1} = 2u_0u_1 + u_1^2$$

$$\overline{A_2} = 2u_0u_2 + 2u_1u_2 + u_2^2$$

$$\overline{A_3} = 2u_0u_3 + 2u_1u_3 + 2u_2u_3 + u_3^2$$

$$\overline{A_4} = 2u_0u_4 + 2u_1u_4 + 2u_2u_4 + 2u_3u_4 + u_4^2$$

When $\boldsymbol{f(u) = u^3}$ the polynomials can be computed by:

Using formula (3):

$$A_0 = u_0^3$$

$$A_1 = 3u_0^2u_1$$

$$A_2 = 3u_0^2u_1 + 3u_1^2u_0$$

$$A_3 = u_1^3 + 3u_0^2u_3 + 6u_0u_1u_2$$

$$A_4 = 3u_0^2u_4 + 3u_1^2u_2 + 3u_2^2u_0 + 6u_0u_1u_3$$

Using (El-Kalla formula) formula (4):

$$\overline{A_0} = u_0^3$$

$$\overline{A_1} = u_1^3 + 3u_0^2u_1 + 3u_1^2u_0$$

$$\overline{A_2} = u_2^3 + 3u_0^2u_2 + 3u_1^2u_2 + 3u_2^2u_0 + 3u_2^2u_1 + 6u_0u_1u_2$$

$$\overline{A_3} = u_3^3 + 3u_0^2u_3 + 3u_1^2u_3 + 3u_2^2u_3 + 3u_3^2u_0 + 3u_3^2u_1 + 3u_3^2u_2 + 6u_0u_1u_3 + 6u_1u_2u_3 + 6u_0u_2u_3$$

V. Numerical example's

5.1. Example (1)

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Consider the following nonlinear volterra integro-differential equation with the exact solution is

$u(x) = e^{x}$,

$$u'(x) = \frac{3}{2}\, e^{x} - \frac{1}{2} e^{3x} + \int_{0}^{x} e^{x-t}\, u^{3}(t)\, dt\,, \qquad u(0) = 1$$

After solving this example by using Adomian formula and El-Kalla formula (two iterations) we compare the result with the exact solution.

Table 1: The Absolute Relative error (ARE) between the Exact solution and Solution using El-Kalla polynomials, also between the Exact solution and Solution using Adomian polynomials for some values of x in Example (1).

X	(ARE) of solution using Adomian polynomials	(ARE) of solution using El-Kalla polynomials
0.1	5.0058×10^{-8}	1.7150×10^{-8}
0.2	4.2724×10^{-6}	1.5113×10^{-6}
0.3	6.4738×10^{-5}	2.3772×10^{-5}
0.4	4.8175×10^{-4}	1.8476×10^{-4}
0.5	0.0024	9.7492×10^{-4}
0.6	0.0094	0.0040

Table 1 shows The Absolute Relative Error (ARE) between the exact solution and the solution using Adomian polynomials. Also, the (ARE) between the exact solution and the solution using El-Kalla polynomials.

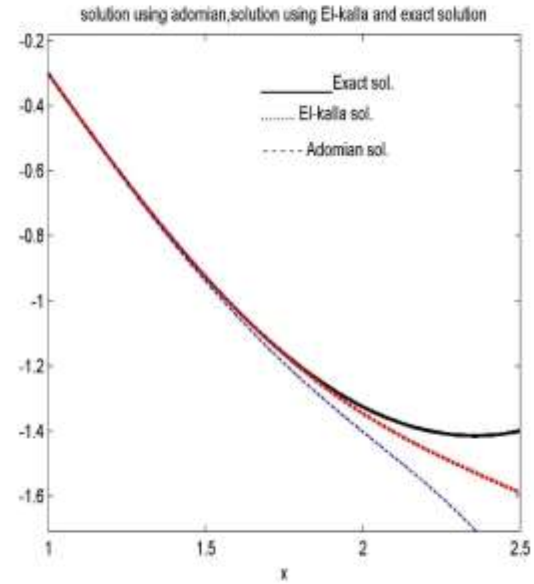

Figure 1. Solution using Adomian polynomials, Solution using El-Kalla polynomials and Exact solution of example (1).

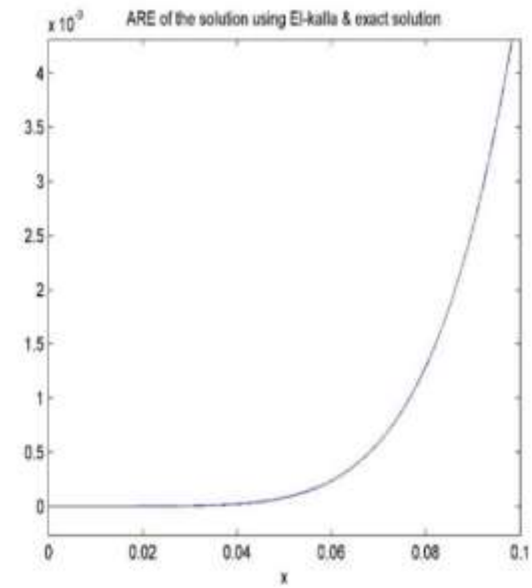

Figure 2. The Absolute Relative Error (ARE) between Exact solution and Solution using El-Kalla polynomials of example (1).

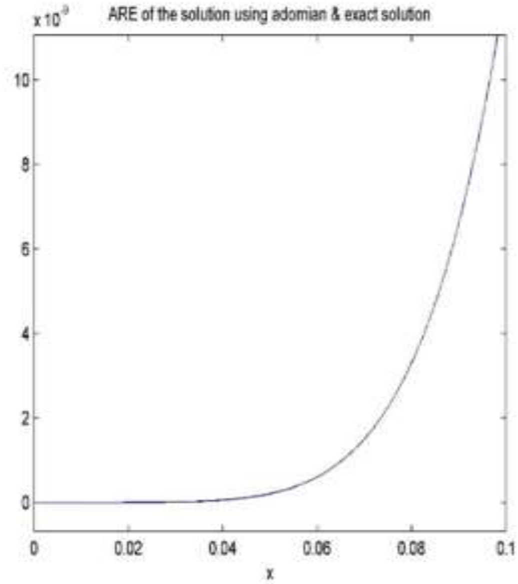

Figure 3. The Absolute Relative Error (ARE) between Exact solution and Solution using Adomian polynomials of example (1).

5.2. Example (2)

Consider the following nonlinear volterra integro-differential equation with the exact solution is

$u(x) = cos\, x - sin\, x,$

$$u' = -2\, sin\, x - \frac{1}{3} cos\, x - \frac{2}{3} cos\, 2x + \int_0^x cos(x-t)\, u^2(t) dt, \quad u(0) \quad = 1.$$

After solving this example by using Adomian formula and El-Kalla formula (two iterations) we compare the result with the exact solution.

Table 2: The Absolute Relative error (ARE) between the Exact solution, Solution using Adomian polynomials, solution using El-Kalla polynomials and solution using Laplace-Adomian modification for some values of *x* in Example (2).

X	(ARE) of solution using Adomian polynomials	(ARE) of solution using El-Kalla polynomials	(ARE) of solution using Laplace-Adomian modification

0.1	1.2315×10^{-8}	4.7754×10^{-9}	0.0012
0.2	6.9047×10^{-7}	2.5802×10^{-7}	0.0044
0.3	6.8092×10^{-6}	2.4322×10^{-6}	0.0089
0.4	3.2741×10^{-5}	1.1057×10^{-5}	0.0139
0.5	1.0569×10^{-4}	3.3246×10^{-5}	0.0148
0.6	2.6423×10^{-4}	7.5844×10^{-5}	0.0221
0.7	5.5253×10^{-4}	1.4058×10^{-4}	0.0245

Table 2 shows The Absolute Relative Error (ARE) between the exact solution and the solution using Adomian polynomials, The Absolute Relative Error (ARE) between the exact solution and the solution using Laplace-Adomian modification [20] and the (ARE) between the exact solution and the solution using El-Kalla polynomials.

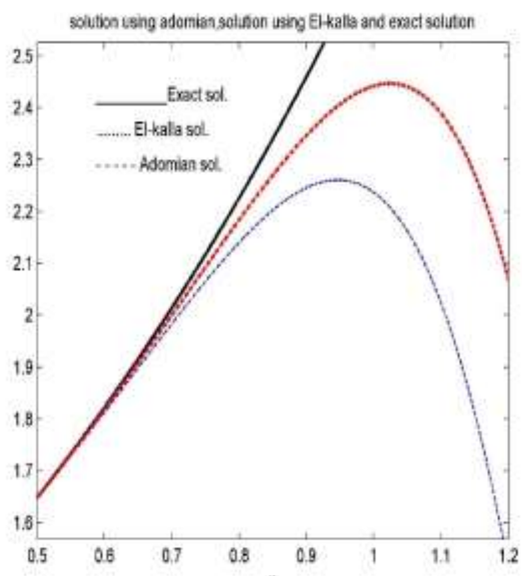

Figure 4. Solution using Adomian polynomials, Solution using El-Kalla polynomials and Exact solution of example (2).

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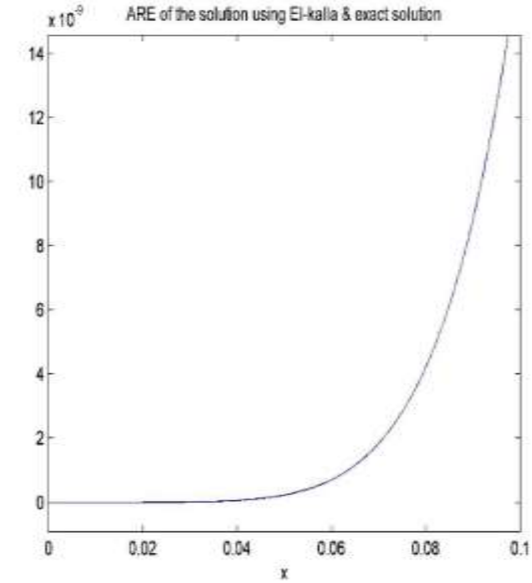

Figure 5. The Absolute Relative Error (ARE)between Exact solution and Solution using El-Kalla polynomials of example (2).

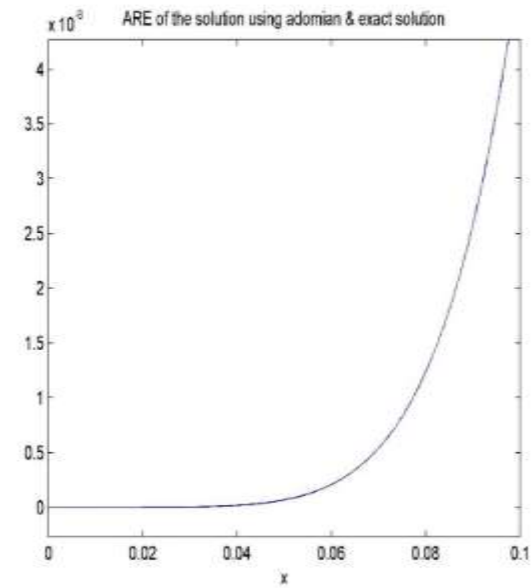

Figure 6. The Absolute Relative Error (ARE) between Exact solution and Solution using Adomian polynomials of example (2).

5.3. Example (3)

Consider the nonlinear Volterra integro-differential equation

$$u' = -\frac{e^{-9x}}{6} + \frac{e^{-3x}}{6} - e^{-x} + \int_{0}^{x} e^{-(x+t)}\, u^2(t)\, dt \,,$$

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$$0 \le t \le x \le 1 .$$

we calculate the maximum truncation error of the accelerated formula without solving the equation as shown in table 3.

Table 3 : shows the maximum truncation error $\Delta^* = \frac{K\,\alpha^{(m+1)}}{L(1-\alpha)}$ for different values of $m,$ where m is the number of Adomian polynomials, μ=1, T=1, M=e^2 , L=2,$\alpha = \frac{2}{e^2}$ and K=0.36.

m	$\Delta^* = \frac{K\alpha^{(m+1)}}{L(1-\alpha)}$
5	9.7×10^{-5}
10	1.4×10^{-7}
15	2.04×10^{-10}
20	2.975×10^{-13}

VI. Conclusion

The main idea of this work was to give a simple Adomian polynomials formula for solving the integro-differential equations. we have introduced El-Kalla formula which gives a new Adomian polynomials for solving nonlinear integro-differential equations. The method gives approximate solutions iteratively with a smaller number of computational steps. The results reveal that the proposed method is simple to execute, effective and it is a recursive relation. When using El-Kalla polynomials the solution becomes easier, faster, programmable and accurate than using the Adomian polynomials and the combined Laplace-Adomian method. also, we can calculate the maximum truncation error without solving the integro-differential equations according to its type.

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