ANALYSING THE COMMUTING GRAPHS FOR ELEMENTS OF ORDER 3 IN MATHIEU GROUPS

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ABSTRACT-- *Assume that G is a finite group and X is a subset of G. The commuting graph is denoted by C(G,X) which has a set of vertices X with two distinct vertices x, y $\in$ X being connected together on the condition of xy = yx. In this paper, computational approaches applied to investigate the structure of commuting graphs C(G,X) when G is one of the Mathieu groups along with X a G-conjugacy class for elements of order 3. We will pay particular attention to analyze the discs structure and determinate the diameters, girths and the clique number for these graphs.*

Keywords-- *Sporadic groups; commuting graph; diameter, cliques.*

MATHEMATICS SUBJECT CLASSIFICATION 2010: 20D08,05C25 ,05C69.

I. INTRODUCTION AND PRELIMINARIES

It is believed that studying the action of a group on a graph is one of the best comprehensible ways of analyzing the structure of the group. Suppose that G is a group and X is a subset of G; the commuting graph is denoted by $\mathcal{C}$(G,X) which has the set of vertices X with two vertices x, y $\in$ X are connected if x $\neq$y, where xy = yx. The commuting graphs were first illustrated by Fowler and Brauer in the seminal paper [1], they were eminent for giving evidence of a prescribed isomorphism of an involution centralizer, where there is a limited number of non-abelian groups capable of containing it. These graphs are extremely vital for the works of the Margulis-Platanov conjecture (see [2] as the graphs mentioned in [1] have X = G\{1} where 1 is the identity element of G). When X is a conjugacy class of involution, the commuting graph known as the commuting involution graph. Rowley, Hart (nèe Perkins), Bates, and Bundy put their efforts into investigating the commuting involution graphs and supplying the diameters and disc sizes (see [3,4,5 and 6]). Suppose that X a conjugacy class of elements of order 3, Nawawi and Rowley in [7] analyzing the C(G,X) when G is either a symmetric group S_n or a sporadic group McL. The aim of this paper is to investigate the commuting graphs when G is one of the Mathieu groups along with X a G-conjugacy class for elements of order 3. The research involves scrutinizing the discs structures and calculating the diameters, girths and the clique number for these graphs. From now, we shall assume that G is one of the aforementioned groups. Also, we let t be an element of order 3 in G and X = t^G. Clearly G, acting by conjugation, induces graph automorphisms of $\mathcal{C}$(G,X) and is transitive on its vertices. For x $\in$ X and i $\in \mathbb{Z}^+$; $\Delta_i(x)$ denotes the set of vertices of $\mathcal{C}$(G,X) which has distance i from x,using the usual distance function for graphs, this distance function will be denoted by d(;). We use G_x(= $C_G(x)$) to denote the stabilizer in G of x. Obviously $\Delta_i(x)$ will be a

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Received: 12 Mar 2020 | Revised: 22 Apr 2020 | Accepted: 09 May 2020

union of certain Gx -orbits. Therefore, we are looking for finding the G_x-orbits of X. In computational group theory Magma [8] and GAP [9] packages are considered the most commonly utilized. In most steps of the algorithm Gap will be dominant in the implementation, while the permutation character of the centralizer of t in G ($C_G(t)$) may be verified using magma and hence the number of $C_G(t)$-orbits (Permutation Rank on X) under the action of X on $C_G(t)$ is calculated. Finally, we will use the online Atlas of Group Representations [10] to get a class name of the groups and we refer to it as The Online Atlas.

For the aforesaid groups the sizes of conjugacy classes for elements of order 3 and the permutation ranks on each class. Also, the structure of the centralizer of t in G which can be seen in [11] are given in the next table.

Table 1. Disc Sizes and Permutation Character

Group	Class	Size of Class	Permutation Rank	$C_G(t)$
M_{11}	3A	440	30	$C3 \times S3$
M_{12}	3A	1760	44	$((C3 \times C3) : C3) : C2$
M_{12}	3B	2640	87	$C3 \times A4$
M_{22}	3A	12320	364	$C3 \times A4$
M_{23}	3A	56672	356	GL(2, 4)
M_{24}	3A	226688	272	C3.A6
M_{24}	3B	485760	1018	$C3 \times PSL(3,2)$

II. COMPUTATIONAL METHOD

Let $x \in \Delta_i(t)$ and $z \in C_G(t)$ one can see immediately that $x^z \in \Delta_i(t)$. Thus for a finite group G, each disc $\Delta_i(t)$ of the commuting graph $\mathcal{C}(G,X)$ is a union of specific $C_G(t)$-orbits.

The size of $C_G(t)$ -orbits under the action of conjugation on t^G can be calculated by using the character table of the group, as we can see in the following result:

Proposition 2.1. [12] Let G be a group acting transitively on a finite set Ω, with a permutation character χ. Suppose that $\alpha \in \Omega$ and that G_α has exactly k orbits on Ω. Then $<\chi,\chi> = k$.

The quantity k in **Proposition 2.1** is called the permutation rank of G_α on Ω .Therefore, the permutation rank of $C_G(t)$ on X is the number of CG(t)-orbits under the conjugation action on X.

Now, let C be a G-conjugacy class. It is obvious that the set $XC = \{x \in X : tx \in C\}$ under the conjugation action of CG(t)-breaks up into sub orbits. Thus to find all the sub orbits of X, we have to identify the CG(t)-orbits of XC, for all those C such that $X_C \neq \emptyset$.

Let C_i , C_j and C_k be the conjugacy classes of a finite group G. Then for a fixed element $g \in C_k$, define the set $a_{ijk} = |\{(g_i, g_j) \in C_i \times C_j \mid g_i g_j = g\}|$.

Then for all possible i, j, k the value a_{ijk} is called a class structure constant for G.

The next lemma will be used to compute the class structure constants for G

Lemma 2.3. [13] Let G be a finite group with n conjugacy classes $C_1, C_2, \ldots, C_n$. Then for all i, j, k we have g_i

International Journal of Psychosocial Rehabilitation, Vol. 24, Issue 08, 2020
ISSN: 1475-7192

$$a_{ijk} = \frac{|G|}{|C_G(gi)||C_G(gj)|} \Sigma_{\chi \in Irr(G)} \frac{\chi(gi)\chi(gj)\overline{\chi(gk)}}{\chi(I)}$$

Where g_i , g_j and g_j are respectively in C_i,C_j and C_k, and $\chi \in Irr(G)$ be the irreducible character table in G.

We should not that $|X_C| = |\{(c, x) \in C \times X \mid cx = t\}| = |a_{ijk}|$. Then by employing **Lemma 2.3** we get

$$|X_C| = \frac{|G|}{|C_G(g)||C_G(t)|} \quad \Sigma_{\chi \epsilon Irr(G)} \frac{\chi(g)\chi(t)^2}{\chi(I)}$$

Therefore, from the complex character table of G, which is available in GAP character table library, and using the GAP function "Class Multiplication Coefficient" we immediately obtain the size of X_C.

III. DIAMETERS, GIRTHS AND CLIQUES NUMBER

To determinate the girths and the cliques number for the C(G,X) we will use the fact that the graph is regular to generated the graph by using the gap package **YAGS** [14] (specifically **GraphByRelation**) on the connected component which contains t. This will provide us the Girths and Cliques Number. in the next algorithm which can be realized by using the definition of the commuting graph. The algorithm is given as follows:

Algorithm 1
Input: The group G, $t \in G$ (the elements of order 3);
i: $X \leftarrow t^G$: the G-conjugacy class of t meet the centralizer in G of t. **ii**: Rel ← {the set of elements satisfies the condition : $x \neq y$ and $x*y = y*x$} **iii**: *C* (G,X) ←**GraphByRelation**(Rel,X). **v**: Grith ←Girth(C (G,X)) & ω(*C* (G,X))←Clique number (C (G,X)) .
Output: Girth and the clique number of C (G,X).

The next table yields by applying **Algorithm 1** on the group in **Table** 1:

Table 2. Girth and cliques number			
Graph	Girth	clique number	clique group
$C(M_{11},3A)$	3	8	$C_3 \times C_3$
$C(M_{12}, 3B)$	3	8	$C_3 \times C_3$
$C(M_{12}, 3A)$	3	6	$C_3 \times C_3$
$C(M_{22}, 3A)$	3	8	$C_3 \times C_3$
$C(M_{23}, 3A)$	3	8	$C_3 \times C_3$
$C(M_{24}, 3A)$	3	8	$C_3 \times C_3$
$C(M_{24}, 3B)$	3	6	$C_3 \times C_3$

The last column of the **Table 2** represented the maximum elementary abelian group generated by the maximum clique.

IV. ANALYZING THE DISCS STRUCTURES

This section dedicated to analyzing the structures of the $\Delta_i(t)$ of the commuting graph $C(G, X)$.

4.1. $C_G(t)$-Orbits of $\Delta_i(t)$

The next algorithm employed to break $\Delta_i(t)$ into $C_G(t)$-sub orbits of X_C , for all those C (G-Conjugacy class) such that $X_C \neq \varphi$ and provided their sizes.

Algorithm 2
Input: the group G, $t \in G$, (the elements of order 3), C (G-Conjugacy Class)
i: $X \leftarrow t^G$ the G-conjugacy class of t ii: $C_G(t) \leftarrow$centralizer in G of t. iii: $O \leftarrow$ the orbits in $C_G(t)$ of X. iv: $\|X_C\| \leftarrow$"Class Multiplication Coefficient" of C in X. v: For $i \rightarrow 1$ to size (O) Do vi: If $t * O[i][1]$ Conjugate to $C \rightarrow Y_C \cup O[i]$ vii: Repeat the steps vi, vii until $\|X_C\| = \|Y_C\|$. viii: $X_C \leftarrow X_C = Y_C$. ix: For $j \rightarrow 1$ to size O Do x: If t in $O[j] \rightarrow t = O_j$ (there is only one) xi: $Y_0 \leftarrow Y_0 \cup j$ xii: For i in O[j] Do xiii: For $h \rightarrow 1$ to size (O) Do xiv: If $d(O[h][1], i) = 1 \rightarrow \{Y_1 \cup \{h\}\}\backslash\{Y_0\}$ xv: For x in $Y_1 \rightarrow \Delta_1(t) \cup O[x]$ xvi: for j in Y_1 Do xvii: Repeat the steps x1, x2 and xviii: If $d(O[h][1], i) = 1 \rightarrow \{Y_2 \cup \{h\}\} \backslash\{Y_1 \cup Y_0\}$ xix: For x in $Y_2 \rightarrow \Delta_2(t) \cup O[x]$ xx: Repeat the above steps and replace the Y_{i+1} with Y_i and $\Delta_{i+1}(t)$ with $\Delta_i(t)$.
Output: The positions the sets X_C in the $\Delta_i(t)$ with their sizes.

For each graph in **Table 2** we provide information about the discs structure. We should note that in the next tables the value $n * m$ means the number and the size of $C_G(t)$-orbits in certain $\Delta_i(t)$, respectively. The exceptional case is $C(M_{11},3A)$ as the graph is disconnected:

1-$C(M_{11}, 3A)$: Form **Table 1** permutation rank of 3A is 30 and $C_G(t) \cong C_3 \times S_3$. In **Table 2** one can see that the graph is disconnected and by **Algorithm 2** there are 55 connected 8-components of $C(M_{11}, 3A)$.

2- C(M12, 3A): Form **Table 1** permutation rank of 3A is 44. Form **Table 2** the graph is connected with diameter 6. The centralizer of $t \in 3A$ isomorphic to $((C_3 \times C_3) : C_3) : C_2$. The structure of the $C(M_{12}, 3A)$ can be seen in the following table:

International Journal of Psychosocial Rehabilitation, Vol. 24, Issue 08, 2020
ISSN: 1475-7192

Table 3. Discs structure of C(M_{12}, 3A)

Class Name	$\Delta_1(t)$	$\Delta_2(t)$	$\Delta_3(t)$	$\Delta_4(t)$	$\Delta_5(t)$	$\Delta_6(t)$
1A	1					
2A						9*2
2B						27
3A	6*2					9*,27
3B		18,*2				
4A			27			54
4B			27			54
5A				54*2	54*4	54*2
6A						54*2
6B			27,*2	54*4		
8A					54*4	
8B					54*4	
11A					54	54
11B					54	54

Table 4. Discs structure of *C*(M_{12},3B)

CLASS NAME	$\Delta_1(t)$	$\Delta_2(t)$	$\Delta_3(t)$	$\Delta_4(t)$	$\Delta_5(t)$	$\Delta_6(t)$
1A	1					
2A		12,12				
2B						18
3A	4,4	12,12			12,12	
3B	4,4	12,12		36,36		18
4A				36	36,36	
4B				36	36,36	
5A			36,36	36,36,36 ,36,36,36	18	
6A		12,12		36,36		36,36
6B			36,36,36,36	36,36,36,36	36,36,36,36	
8A				36,36,36,36	,36,36,36,36	
8B				36,36,36,36	36,36,36,36	
10A				36-36-36-36		
11A			36,36,36	36	36,36,36,36	
11B			36,36,36	36	36,36,36,36	

3- $C(M_{12}, 3B)$: Form **Table 1** permutation rank of 3B is 87. Form **Table 2** the graph is connected with diameter 6. The centralizer of t ∊ 3B isomorphic to $C_3 \times A_4$. The structure of the $C(M_{12}, 3B)$ can be seen in the following table:

4- $C(M_{22}, 3A)$: Form **Table 1** permutation rank of 3A is 364. Form **Table 2** the graph is connected with diameter 6. The centralizer of t ∊ 3A isomorphic to $C_3 \times A_4$. The structure of the $C(M_{22}, 3A)$ can be seen in the following table:

Table 5. Discs structure of C(M_{22}, 3A)

CLASS NAME	$\Delta_1(t)$	$\Delta_2(t)$	$\Delta_3(t)$	$\Delta_4(t)$	$\Delta_5(t)$	$\Delta_6(t)$
1A	1					
2A		3, 2*12	36	36	18	18
3A	6*4	6*12,3,2*12	36	7*36	10*36,18	
4A				8*36	6*36	
4B			14*36	6*36	8*36	
5A			14*36	28*36	58*36	
6A		6*12	6*36	8*36		18
7A				31*36	17*36	
7B				31*36	17*36	
8A				8*36	16*36	
11A				8*36	11*36	
11B				8*36	11*36	

Table 6. Discs structure of C(M_{23}, 3A)

CLASS NAME	$\Delta_1(t)$	$\Delta_2(t)$	$\Delta_3(t)$	$\Delta_4(t)$
1A	1			
2A		15*2,60	90	
3A	20*3	60*7,15*2	180,90	180*3,90*4
4A			180*13,,90*2,45	180*6,90*3
5A		60*3	180*15	180*41
6A		60*6	180*16,90*4,45	180*4,90
7A			180*13,90*2	180*21,90*3
7B			180*13,90*2	180*21,90*3
8A			180*8,90*2	180*20,90*2
11A			180*6	180*19
11B			180*6	180*19

5- $C(M_{23}, 3A)$: Form **Table 1** permutation rank of 3A is 356. Form **Table 2** the graph is connected with diameter 4. The centralizer of t ∊ 3Aisomorphic to GL(2, 4). The structure of the $C(M_{23}, 3A)$ can be seen in the following table:

14A			90	180*8
14B			90	180*8
15A		60*3	180*2	180*6
15B		60*3	180*2	180*6
23A			180*4	180*2
23B			180*4	180*2

6-$C(M_{24}, 3A)$: Form **Table 1** permutation rank of 3A is 272. Form **Table 2** the graph is connected with diameter 4. The centralizer of t ∊ 3A isomorphic to $C_3.A_6$. The structure of the $C(M_{24}, 3A)$ can be seen in the following table:

Table 7. Discs structure of $C(M_{24}, 3A)$

CLASS NAME	$\Delta_1(t)$	$\Delta_2(t)$	$\Delta_3(t)$	$\Delta_4(t)$
1A	1			
2A		45-90	135	
2B				18-45
3A	120	360*3,45,90	1080-135	4*180,45,18
3B		360		
4A				135,135
4B			1080*3,540*8,270	1080,270
4C				180*4
5A		360*3	1080*9-	1080*12
6A		360*3	1080*9,540*8-270	1080*2,270,135*2
6B			1080	1080
7A			1080*6--540*6	1080*8,180*3
7B			1080*6--540*6	1080*8,180*3
8A			1080*7-540*2	1080*9,540*2
10A			1080*2	
11A			1080*20	1080*12
12A			540*4	
12B			1080*2	1080*2
14A			1080*4,540*3	1080*4
14B			1080*4,540*3	1080*4

15A		360*3	1080*7	1080*4
15B		360*3	1080*7	1080*4
21A			1080*3	
21B			1080*3	
23A			1080*6	1080
23B			1080*6	1080

6-C(M_{24},3B): Form **Table 1** permutation rank of 3B is 1018. Form **Table 2** the graph is connected with diameter 5. The centralizer of t ∊ 3B isomorphic to $C_3 \times PSL(3, 2)$. The analyzing of the C(M_{24}, 3B) can be seen in the following table:

Table 8. Discs structure of C(M_{24}, 3B)

CLASS NAME	$_1(t)\Delta$	$_2(t)\Delta$	$_3(t)\Delta$	$_4(t)\Delta$	$_5(t)\Delta$
1A	1				
2A			21	42	
2B	56	168,42		126	
3A	56	6*168		168	
3B		5*168,42	504,2*168,21	168,126,42	
4A				2*126	
4B			9*504	2*126	
4C		4*168		8*504	
5A			22*504	4*504,2*168	42,126
6A			23*504,2*168	22*504,168	
6B		2*168	8*504	20*504,4*126	
7A		168*3	5*504	12*504	
7B		168*3	5*504	12*504	
8A			8*504	32*504	
10A			6*504	40*504,4*126	
11A			43*504	89*504	
12A			24*504	36*504	
12B		4*168	56*504	60*504	
14A			32*504	50*504	
14B			32*504	50*504	
15A			26*504	37*504,3*168	
15B			26*504	37*504,3*168	
21A		3*168	16*504	19*504	
21B		3*168	16*504	19*504	

Section 5. Main

The graph C(M_{11},3A) with 55 components as connected involution graph the **Table 2**, the described in theorem:

Theorem the groups of the following

•Diam and $|\Delta_1|$ =13, $|\Delta_2|$, $|\Delta_4|$ =324 , $|\Delta_5|$

•Diam and |Δ1| =17, |Δ2| , |Δ4| =1080 , |Δ5|

•Diam and |Δ1| = 26; |Δ2| 1296; |Δ4| =

Theorem

commuting is disconnected connected 8- seen above. For a commuting C(G, X),given in graph structure following

3.1:. For G one of **Table2** , we have results:

C(M_{12}, 3A) = 6 =36 , $|\Delta_3|$ = 108 =756 $|\Delta_6|$ =522.

C(M12, 3B) = 6 =96 , |Δ3| = 432 =906 |Δ6| =108.

C(M22, 3A) = 6 = 198; |Δ3| =

5184 ,$|\Delta 5|$ = 5580 ,$|\Delta_6|$ = 36.

• Diam C(M23,3A) = 4 and $|\Delta 1|$ = 61; $|\Delta 2|$ = 1440; $|\Delta 3|$ = 20430; $|\Delta 4|$ = 34740.

23A			15*504	9*504	
23B			15*504	9*504	

• Diam C(M24, 3A) = 4 and $|\Delta 1|$ = 123 , $|\Delta 2|$ = 6030, $|\Delta 3|$ = 137970, $|\Delta 4|$ = 82566.

• Diam C(M24; 3B) = 6 and $|\Delta 1|$ = 113, $|\Delta 2|$ = 5796; $|\Delta 3|$ = 191226, $|\Delta 4|$ = 288456, $|\Delta_5|$ = 168.

Proof. Each of $\Delta_i(t)$ of the commuting graph C(G;X) is a union of specific $C_G(t)$- orbits. Thus using the previous tables, we obtain the proof.

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