International Journal of Psychosocial Rehabilitation, Vol. 24, Issue 05, 2020
ISSN: 1475-7192

Nash Equilibrium Strategies in Extensive Form Games and Sub Games

C. Karthi and K. Selvakumari*

Abstract--- *Nash Equilibrium solutions are a common solution concept of extensive form games in a two player zero sum games and bimatrix games. It determines the analysis of sub game perfect pure and mixed strategy Equilibria and sequential Equilibria concept. The main aim of the paper is to investigate the pure and mixed Nash equilibrium strategies of two person zero sum games with extensive form games and sub games and provide tools for its systematic study.*

Keywords--- *Nash Equilibrium Strategies Pure Strategy, Mixed Strategy, Extensive Form Games, Sub Games.*

I. INTRODUCTION

An extensive form game is a specification of a game in game theory, allowing for the explicit representation of a number of key aspects, like the sequencing of players, possible moves, their choices at every decision point, the (possible impact) information each player has about the other players moves when they make a decision, and their payoffs for all possible game outcomes. Some authors, particularly in introductory textbooks, initially define the extensive form game as being just a game tree with payoffs (no imperfect or incomplete information) & add the other elements in subsequent chapters as refinements. This general definition was introduced by Harold W Kuhn in 1953, which extended an earlier definition of Von Neumann from 1928.

In game theory, a sub game is a subset of any game that includes an initial node (which has to be independent from any information set) & all its successor nodes. A sub game perfect equilibrium is a equilibrium not only overall, but also for each sub game, while Nash Equilibria can be calculated for each sub game & also it is a refinement of a Nash equilibrium used in dynamic games. Sub game perfect equilibrium is proposed by Reinhard Selten.

Common method for determining sub game perfect equilibrium in the care of a finite game backward induction. It was first used by Zermelo in 1913, to Prove that chess has pure optimal strategies John Von Neumann & Oskar Morgenstern suggested solving zero sum two person games by backward induction in their Theory of Games & Economic behaviour (1944), the book which established game theory as a field of study. In game theory, a sequential game is a game where one player chooses their action before the others choose theirs. Repeated games are an example of sequential games.

C. Karthi, Research Scholar, Department of Mathematics. E-mail: karthiananya.r@gmail.com
K. Selvakumari, Professor, Department of Mathematics, Vels Institute of Science, Technology and Advanced Studies, Chennai, Tamilnadu, India. E-mail: selvafeb6@gmail.com*

DOI: 10.37200/IJPR/V24I5/PR201701
Received: 07 Feb 2020 | Revised: 01 Mar 2020 | Accepted: 18 Mar 2020

II. DEFINITIONS

2.1. Extensive Form

A detailed description of the sequential structure of the decision problems encountered by the players in a game, often represented as a game tree.

2.2. Perfect Information

All players know the same structure (including the payoff functions at every outcome).

Each player, when making any decision is perfectly informed of all the events that have previously occurred.

2.3. Perfect Information Extensive form Games

A perfect information extensive form game, $G = \{N, H, P, U\}$

Where $N = \{1,2,3 \ldots n\}$ is the set of players.

- H is a set of sequences (Finite or Infinite)
- $h = (a^k)\ k = 1 \ldots .K \in H$ is a History
- $y = (a^k)\ k = 1 \ldots .K \in H$ & $L < K$ then $(a^k)\ k = 1 \ldots \ldots L \in H$
- $(a^k), k = 1$ to $\infty \in H$ is $(a^k)\ k = 1 \ldots .L \in H$ for all positive L
- Z is the set of terminal histories

2.4. Pure Strategies in Perfect Information Extensive form Games

A pure strategy of player $i \in N$ is an extensive form game with perfect information,

G = (N,H,P,U) is a function that assigns an action in A(h) to each Non terminal history

$h \in H/Z$ for P(h)=i

$$A(h) = \{a: (h, a) \in H\}$$

A Pure Strategy is a contingent plan that specifies the action for Player i at every decision node of i.

Note

A pure strategy profile is a weak nash equilibrium, if for all agents i & for all strategies

$$s_i \neq S_i,$$
$$U_i(s_i, S_{-i}) \geq U_i(s_i, S_{-i})$$

2.5. Sub Game

A Sub game G' of an extensive form game G, consists of a single node & all its successors in G, with the Property than if X' $in\ V_G{}'$ & $X'' \in h\ (X')$ then $X'\ in\ V_G{}'$

The information sets & payoffs of the sub game are inherited from the original game.

DOI: 10.37200/IJPR/V24I5/PR201701
Received: 07 Feb 2020 | Revised: 01 Mar 2020 | Accepted: 18 Mar 2020

2.6. Sub Game Perfect Equilibrium

A strategy profile S^k is a subgame perfect Nash equilibrium (SPE) in game G if for any subgame G' of G, $\frac{S^k}{G'}$ is a nash equilibrium of G'.

Note: Every SPE is a NE but not vice versa.

Algorithm

Step 1: To find Nash equilibrium of the smallest game

Step 2: Fix one for each sub game and attach payoffs to its initial node.

Step 3: Repeat with the reduced game.

2.7. Backward Induction Method

Backward induction refers to starting from the last sub games of a finite game, then finding the best response strategy profiles or the Nash equilibria in the sub games, then assigning these strategies profiles & the associated payoffs to be subgames & moving successively towards the beginning of the game (i.e.) plays a sequential rational strategy.

2.8. Sequential rationality

A players equilibrium strategy should specify optimal actions at every point in the game tree.

Theorem 1

Every finite game of perfect information has a pure strategy Nash equilibrium , that can be derived through backward induction. Moreover if no player has the same payoffs at any two terminal nodes, then backward induction results in a unique Nash equilibrium.

Theorem 2

Backward induction given the entire set of SPE.

Theorem 3

Every finite perfect information extensive form game G has a pure strategy SPE, that can found by backward induction.

Theorem 4

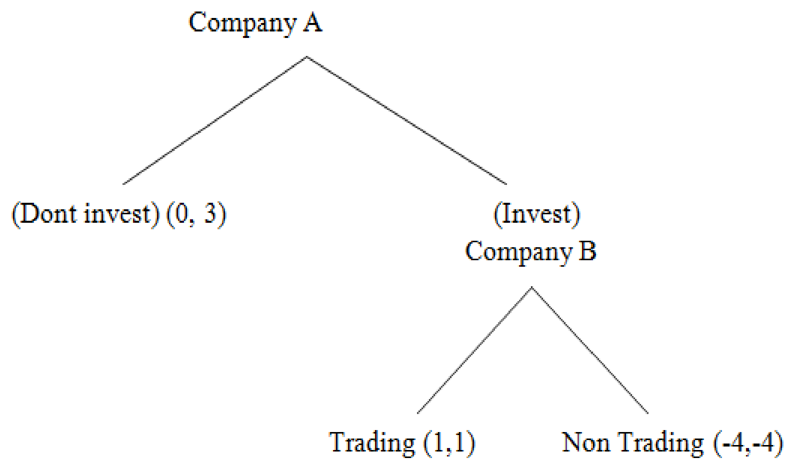
Every finite extensive form games G has a SPE.

III. BEST RESPONSE

Numerical Example 1

Stock market investment problem (with perfect information)

DOI: 10.37200/IJPR/V24I5/PR201701
Received: 07 Feb 2020 | Revised: 01 Mar 2020 | Accepted: 18 Mar 2020



The normal form representation is given as

		Company (B)	
Company (A)		Trading	Non Trading
	Invest	(1,1)	(-4,-4)
	Dont invest	(0,3)	(0,3)

Therefore (0,3) and (1,1) are the pure strategy Nash equilibrium. Similarly to find the mixed strategy Nash equilibrium.

Company (A)		Company (B)	
	(1,1)	(-4,-4)	p_1
	(0,3)	(0,3)	p_2
	q_1	q_2	

$$\in (A) \Rightarrow q_1 - 4q_2 = 0 - \quad ①$$

$$\in (A) \Rightarrow 0q_1 - 0q_2 = 0 - \quad ②$$

Solving ① & ②

$$q_1 - 4q_2 = 0$$

$$q_1 = 4q_2$$

Since $q_1 + q_2 = 1$

$$4q_2 + q_2 = 1$$

$$5q_2 = 1$$

$$q_2 = 1/5$$

$$q_1 = 4/5$$

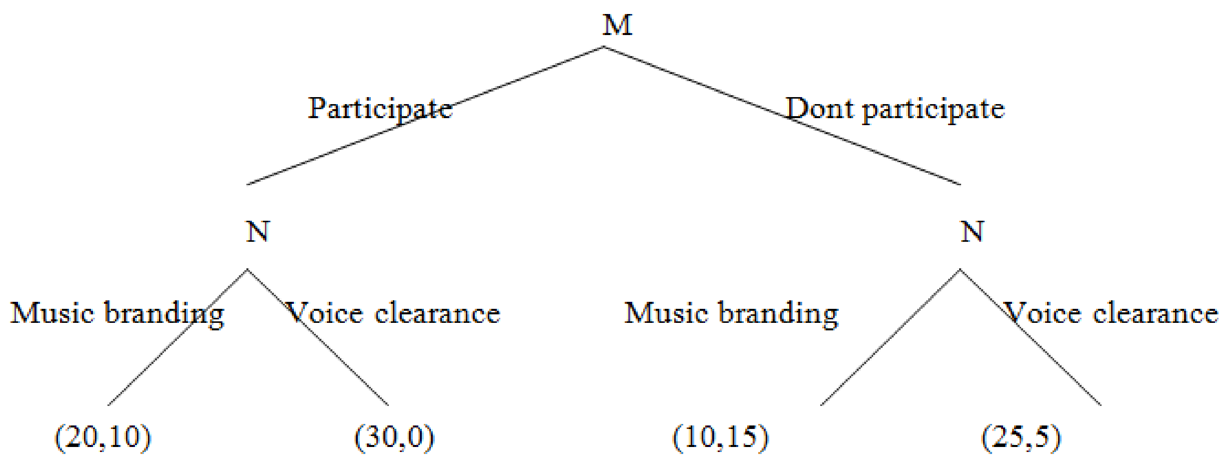
(4/5,1/5) → Mixed Nash equilibrium

Numerical Example 2

Solving Sequential & Simultaneous Move Games Using Backward Induction Method

Music contests: Musicians M &N are unaware about the strategy of each other. Both of them work on the perception that the other one would adopt the best strategy for itself.

DOI: 10.37200/IJPR/V24I5/PR201701
Received: 07 Feb 2020 | Revised: 01 Mar 2020 | Accepted: 18 Mar 2020

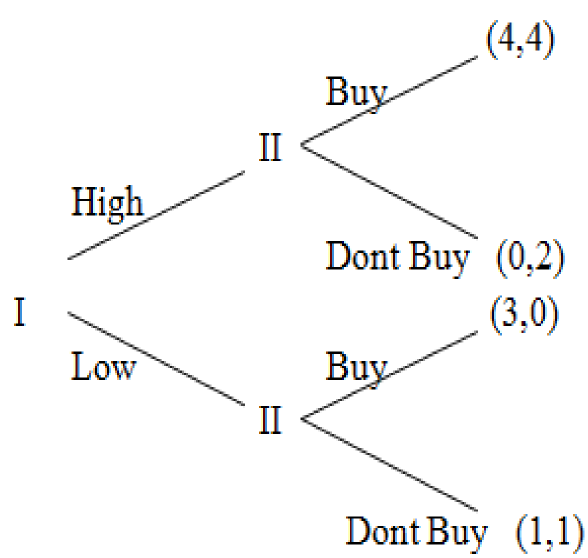


Using backward induction method verifying best response, to the payoffs (20,10) is the only pure Nash equilibrium point.

		N	
		MB	**VC**
M	p_1	(20,10)	(30,0)
	p_2	(10,15)	(25,5)

Numerical Example 3 -Backward induction

Player I acts an internet service provider and player II a potential customer. They consider entering into contact of service provision for a period of time. The provider decides between two action to buy or not to buy. The service provider , Player I makes the first move, choosing high or low quality of service. Then the customer player II is informed about the choice Player II can then decided separately between buy and don't buy.

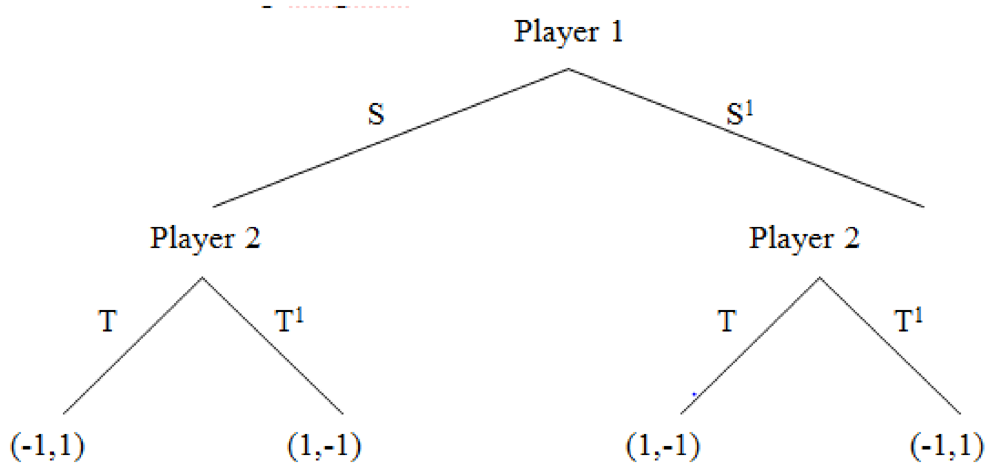


By Backward Induction method, (4,4) is the only Pure Nash Equilibrium

DOI: 10.37200/IJPR/V24I5/PR201701
Received: 07 Feb 2020 | Revised: 01 Mar 2020 | Accepted: 18 Mar 2020

Numerical Example 4 -Sub game Perfect Nash Equilibrium

Consider the following sub game



Player 2

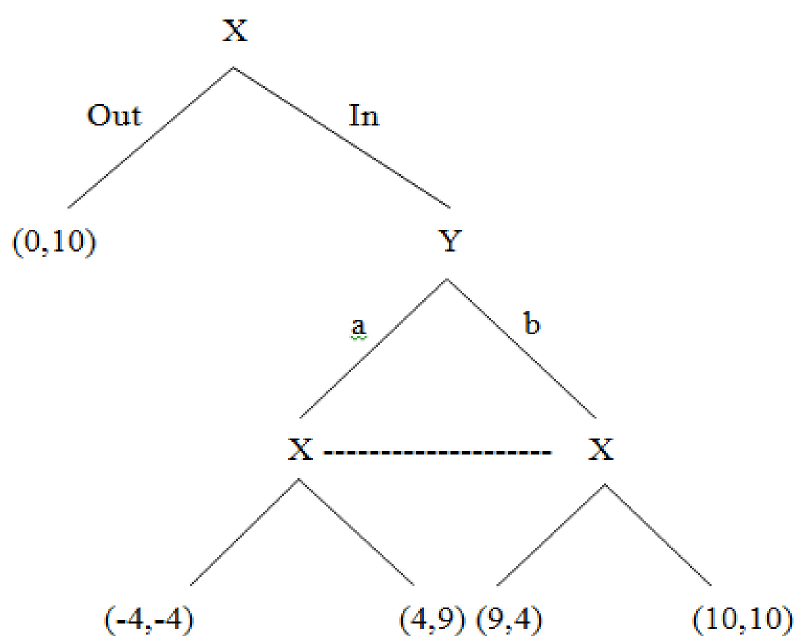
		T	**T^1**
Player 1	**S**	(-1,1)	(1,-1)
	S^1	(1,-1)	(-1,1)

By Backward Induction method, (-1,1) is the Pure SNE

In this game, there are 2 proper subgames & the game itself which is also a subgame & thus a total of 3 sub games.

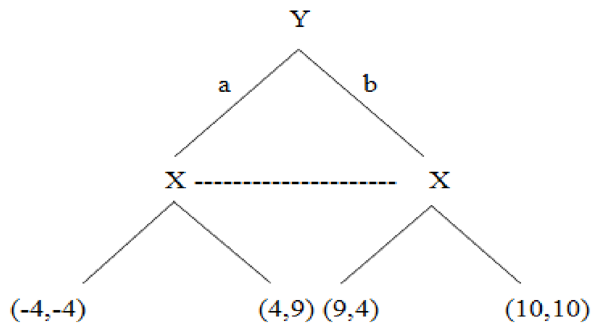
Numerical Example 5

Consider the following game with imperfect information.



DOI: 10.37200/IJPR/V24I5/PR201701
Received: 07 Feb 2020 | Revised: 01 Mar 2020 | Accepted: 18 Mar 2020

The smallest sub game is

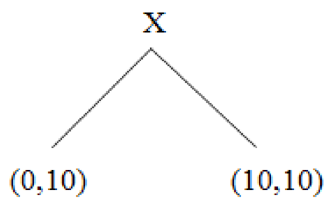


& its strategic form is

		X	
		a	**b**
X	**a**	(-4,-4)	(9,4)
	b	(4,9)	(10,10)

Nash Equilibrium of the Sub game is (b,b) is (i.e.) (10,10)

The Reduced Sub game is

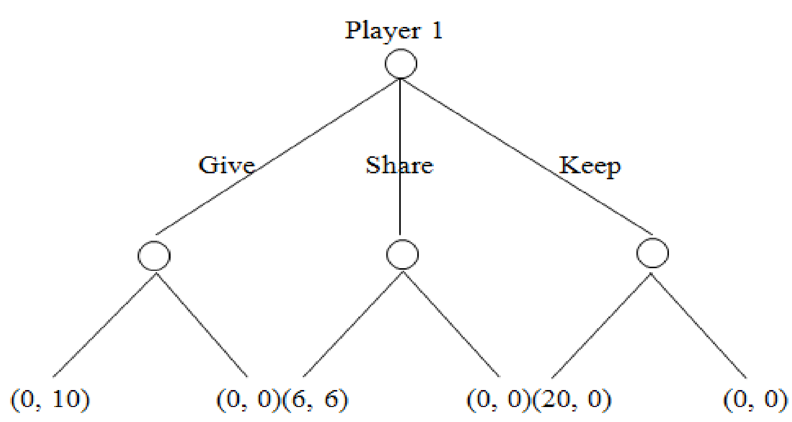


Therefore (10,10) is the one and only sub game perfect equilibrium for the game.

Numerical Example 6

There are 2 players, Player 1 has 10 dollar, She can choose togive (10 dollars to Player 2 & 0 to Player 1), share (5 dollars to each player) or keep (10 dollars to Player 1 & 0 dollars to Player 2). After she makes her decision, which Players 2 observes, Player 2 can accept or reject. After accepting payoffs are as specified (lets assume utilities are dollar amounts) & after rejecting everyone gets 0.

Solution: The extensive form representation for the same is



By using backward induction method, (6,6) & (20,0) → two sub game perfect equilibrium.

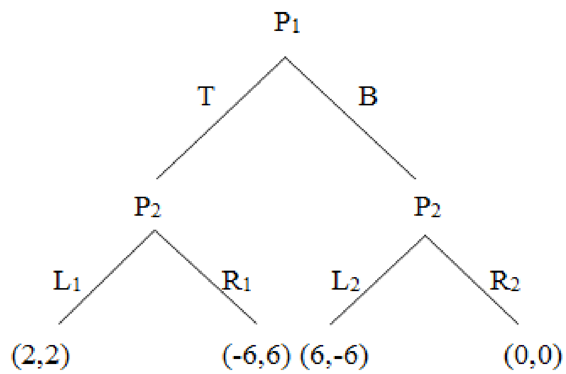
In Normal form representation we have to describe all the complete strategies available to each of the players. For Player 2, let AG mean, that Player 2.

Player A if Player 1 Gives, RG mean that Player 2 plays R if Player 1 Giver & so on. In this case, we will yield a 3x8 matrix

	(AG,AS,AK)	**(AG,AS,RK)**	**(AG,RS,AK)**	**(AG,RS,AK)**	**(RG,AS,AK)**	**(RG,AS,RK)**	**(RG,RS,AG)**	**(RG,RS,RK)**
G	(0,20)	(0,20)	(0,20)	(0,20)*	(0,0)	(0,0)	(0,0)	(0,0)
S	(6,6)	(6,6) *	(0,0)	(0,0)	(6,6)	(6,6) *	(0,0)	(0,0)
K	(20,0) *	(0,0)	(20,0)*	(0,0) *	(20,0) *	(0, 0)	(20,0) *	(0,0) *

* → Nash Equilibria of the game

Numerical Example 7



Totally 3 subgames

Norm form game is

		P₂			
		L_1 L_2	**L_1 R_2**	**R_1 L_2**	**R_1 R_2**
P_1		(2,2)	(2,2)	(-6,6)	(-6,6)
		(6,-6)	(0,0)	(6,-6)	(0,0)

The Pure Strategy Nash equilibrium is (0,0) (i.e.) (B_1 (R_1 R_2))

Finding mixed strategy for the same, by Iterative elimination of dominance method,

(0,0) is the one and only Mixed Strategy Nash Equilibrium.

IV. Concluding Comments

This study proposed some methods to solve a two-person extensive form games for pure mixed strategy. The proposed methodology is quite general & also it can be applicable to sub games too. Therefore this research can be expanded to Multiplayer games. As this study has reviewed only a tiny portion of the extensive form games, there remains a need for a study of Equilibria nature under uncertainty.

REFERENCES

[1] Dresher M (1961). The Mathematics of games of strategy. Theory & applications (Ch 4: Games in Extensive form, PP 74-78) *Rand Corp ISBN* 0-486-64216

[2] J. Kuipers, J. Flesch, G. Schoenmakers, K. Vrieze, 2009, European Journal of Operational Research, Pure subgame – *Perfect Equilibria in Free transition games.*

[3] Refinement of the Nash equilibrium concept by E Van Damme.

[4] K G Binmore, 1985, *Equilibria in Extensive form games The Economic Journal* 95 (supplement), 51-59.

[5] Kuhn H 1950 Extensive games, *Proceedings of the National Academy of Sciences of the United States of America* 36, 570-6

[6] Kuhn H 1953 Extensive games the Problem of information In Contributions to the Theory of Games, Volume II (Annals of Mathematics studies 28) *Ed. H Kuhn & A Tucker Princeton*

[7] Kimmo Berg, Gijs Schoenmakers 2017, Construction of Subgame – *Perfect Mixed Strategy Equilibria in Repeated games* 8 (4), 47

[8] Nash JF (1951), Non Co-operative gamer, *Ann of Math,* Vol 54, PP 286-295.

[9] Philip J Reny 1992, *The Journal of Economic Perspectives Rationality in Extensive form games,* Vol 16, No 4, PP 103-118

[10] R J Aumann, *Game Theory*

[11] Reny P 1999 on the existence of Pure & Mixed Strategy Nash Equilibrium in discontinuous games. *Econometrica* 67, 1029-56

[12] Robert Stalmaker, 1999, Research in Economics; Extensive & Strategic forms Games & Models for Games Vol 53, Issue 3, PP 293 –319

[13] Selten Rein (1975), Re-examination of the Perfectness concept for equilibrium points in extensive form games. *Int J Game Theory,* Vol 4, PP 25-55.

[14] Von Neumann J & Morgenstern .o 1944 Theory of Games & Economic Behaviour. *Princeton, NJ Princeton University Press.*

DOI: 10.37200/IJPR/V24I5/PR201701
Received: 07 Feb 2020 | Revised: 01 Mar 2020 | Accepted: 18 Mar 2020