A Single Server Non-Markovian queue with K Phase of Vacation, Two Types of Services and an Optional Service

V. Suvitha and R. Kalyanaraman

Abstract--- *A single server queue with two types of services and with vacation has been considered. The type 1 service is a phase type service with two service phases. Both the service time distributions are generally distributed. The type 2 service has only one phase of service. In addition the server also provides an optional service. This service time distribution is also general. At each time the system becomes empty, the server takes K phases of vacation and the vacation time distribution of each phase is general. For this model the probability generating function for the number of customers in the queue at different server's state are obtained using the supplementary variable technique. Some performance measures and particular model are calculated and numerical results are presented.*

Keywords--- *Phases Service – Optional Service – Supplementary Variable Technique –Vacation – Performance Measures.*

AMS Subject classification number: 90B22, 60K25 and 60K30.

I. Introduction

Queues with vacation to server has applications in real life. Some examples are Production system, Bank service, Computer and communication system. In the $M/G/1$ queuing system, the concept of vacation had been first studied by Keilson and Servi (1987), they introduced the concept of modified service time which has a main role in the system with general service and vacation times. The classic $M/G/1$ queue with various vacation policies have been studied by several researchers, including Doshi (1986,1990), Gross and Harris (1998), Ke (1986), Takagi (1991). The two monographs of Tian and Zhang (2006) and Takagi (1991) collected the research results of the $M/G/1$ vacation queues. Chen et al. (2009) considered a $GI/M/1$ queue with phase type of working vacation and vacation interruption where the vacation time follows a phase type distribution. Tian and Zhang (2001) treated the Geo$/GI/1$ system with variant policies. In this system, they assumed that after serving a customer in the system the server take a random number of vacation before returning to service station. Tian and Zhang (2002, 2003) discussed the discrete time $GI/Geo/1$ queue with server vacations and the $GI/M/1$ queue with PH vacations or setup times, respectively. Ke and Chu (2006) analyzed the $M^{[x]}/G/1$ queue with mthe modified vacation policyby stochastic decomposition property and Ke (2007) used supplementary variable technique to study an $M^{[x]}/G/1$ queue with balking under a variant vacation policy.

Ke (2003) made the contribution to the control policy of $M/G/1$ queue with server vacations, startup and

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breakdown. He obtained the system characteristic of the model and obtained the total expected cost function per unit time to determine the optimal threshold of N policies at a minimum cost. Ke et al. (2010), studied the operating characteristics of an $M^{[x]}/G/1$ queueing system with N-policy and at most J vacations. In this model they assumed that the server takes at most J vacations repeatedly until at least N customers returning from a vacation are waiting in the queue.

In this paper, we consider an M/G/1 queue with server takes K phases of vacation, when there are no customers in the system and the server provides two type of services called essential(phase) and an optional service to the customer. The paper is organized as follows: The corresponding mathematical model is defined in section 2 and the governing differential difference equations, the boundary conditions and the normalizing condition are given in section 3. For this model the probability generating function of the number of customers in queue irrespective of the server state is derived in section 4. Also, some performance measures related to this queueing model are derived from these probability generating functions and are given in section 5. Some particular models are given in section 6 and in section 7 a numerical study is also carried out. Last section presents a conclusion.

II. THE MODEL

The arrival follows Poisson with rate $\lambda(>0)$ and a single server provides two type of services, respectively called type 1 service and type 2 service. Also the server provides an optional service. The entering customers selects type 1 service with probability p (0<p<1) or type 2 service with probability $1-p$. The type 1 service is a phase type service with two phases. After completion of type 1 service or type 2 service, the customer leaves the system with probability $1-r$ (0<r<1) or choose an optional service with probability r. The service time distributions are, generally, the distribution functions are $B_{1,l}(x)$, for type 1 and l^{th} phase of service $(l=1,2)$, $B_{2,1}(x)$, for type 2 service, $B_{2,2}(x)$, for optional service. The Laplace- Stieltjes transform (LST) for $B_{i,l}(x)$ is $B^{*}_{i,l}(\theta)$ and finite k^{th} moments are $E(B^{k}_{i,l}), k\geq 1, i,l=1,2.$

After completion of service if there are no customers in the system, the sever takes K phases of vacation. The jth phase of vacation time V_j follows a general distribution with distribution function $V_j(x)$, Laplace- Stieltjes transform (LST) $V^{*}_{j}(\theta)$ and finite moments $E(V^{k}_{j})$, $1\leq j\leq K, k\geq 1.$

It may be noted that $B_{i,l}(x), V_j(x), (B_{i,l}(\infty)=1, B_{i,l}(0)=0, V_j(\infty)=1, V_j(0)=0)$ are continuous, so that $\mu_{i,l}(x)dx, (\gamma_j(x)dx)$ are the first order differential functions (hazard rates) of $B_{i,l}(x), (V_j(x))$, $i,l=1,2,$ $1\leq j\leq K.$

For the analysis the supplementary variable (the variable is elapsed time) technique has been used.

Let $\mu_{1,l}(x)dx$ be the conditional probability of completion of the l^{th} phase of type 1 service during the interval

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$(x, x+dx],$ given that the elapsed service time is x so that $\mu_{1,l}(x) = \dfrac{b_{1,l}(x)}{1 - B_{1,l}(x)}, (l = 1,2)$ and let $\mu_{2,1}(x)dx(\mu_{2,2}(x)dx)$ be the conditional probability of completion of the type 2 service (optional service)during the interval $(x, x+dx],$ given that elapsed service time is x so that $\mu_{2,1}(x) = \dfrac{b_{2,1}(x)}{1 - B_{2,1}(x)}(\mu_{2,2}(x) = \dfrac{b_{2,2}(x)}{1 - B_{2,2}(x)})$ and let $\gamma_j(x)dx$ be the conditional probability of completion of the jth phase of vacation during the interval $(x, x+dx],$ given that the elapsed vacation time is x so that

$$\gamma_j(x) = \frac{v_j(x)}{1 - V_j(x)}, 1 \le j \le K.$$

The following notations have been introduced to define the model mathematically:

$P_n^{(1,l)}(x,t) = \Pr\{$at time t, there are n customers in the queue excluding one in lth phase of type 1 service and the elapsed service time is $x\}$, $l = 1, 2, n \ge 0$,

$P_n^{(2,1)}(x,t) = \Pr\{$at time t, there are n customers in the queue excluding one in the type 2 service and elapsed service time is $x\}, n \ge 0$,

$P_n^{(2,2)}(x,t) = \Pr\{$at time t, there are n customers in the queue excluding one in the optional service and elapsed service time is $x\}, n \ge 0$,

$V_n^{(j)}(x,t) = \Pr\{$at time t, the server is on jth phase of vacation with elapsed vacation time is x and the number of customers in the queue is $n\}, 1 \le j \le K, n \ge 0$ and

$Q(t) = \Pr\{$at time t, there are no customers in the system and the server is idle$\}$.

Let $P_n^{(i,l)}(x)\ (i, l = 1,2),\ V_n^{(j)}(x)$ and Q denote the corresponding steady state probabilities.

The probability generating functions (p.g.f's) for the probabilities $\{P_n^{(i,l)}(x)\}$, $(i, l = 1,2)$, $\{V_n^{(j)}(x)\}$, $1 \le j \le K$ are respectively defined as

$$P^{(i,l)}(x,z) = \sum_{n=0}^{\infty} z^n P_n^{(i,l)}(x) \text{ and } V^{(j)}(x,z) = \sum_{n=0}^{\infty} z^n V_n^{(j)}(x). |Z| < 1$$

III. THE GOVERNING EQUATIONS

The forward Kolmogorov equations governing the system under steady conditions (Cox(1965)) can be written as follows:

$$\frac{d}{dx} P_0^{(1,1)}(x) + (\lambda + \mu_{1,1}(x)) P_0^{(1,1)}(x) = 0 \tag{1}$$

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$$\frac{d}{dx}P_n^{(1,1)}(x)+(\lambda+\mu_{1,1}(x))P_n^{(1,1)}(x)=\lambda P_{n-1}^{(1,1)}(x);n\geq 1 \tag{2}$$

$$\frac{d}{dx}P_0^{(1,2)}(x)+(\lambda+\mu_{1,2}(x))P_0^{(1,2)}(x)=0 \tag{3}$$

$$\frac{d}{dx}P_n^{(1,2)}(x)+(\lambda+\mu_{1,2}(x))P_n^{(1,2)}(x)=\lambda P_{n-1}^{(1,2)}(x);n\geq 1 \tag{4}$$

$$\frac{d}{dx}P_0^{(2,1)}(x)+(\lambda+\mu_{2,1}(x))P_0^{(2,1)}(x)=0 \tag{5}$$

$$\frac{d}{dx}P_n^{(2,1)}(x)+(\lambda+\mu_{2,1}(x))P_n^{(2,1)}(x)=\lambda P_{n-1}^{(2,1)}(x);n\geq 1 \tag{6}$$

$$\frac{d}{dx}P_0^{(2,2)}(x)+(\lambda+\mu_{2,2}(x))P_0^{(2,2)}(x)=0 \tag{7}$$

$$\frac{d}{dx}P_n^{(2,2)}(x)+(\lambda+\mu_{2,2}(x))P_n^{(2,2)}(x)=\lambda P_{n-1}^{(2,2)}(x);n\geq 1 \tag{8}$$

$$\frac{d}{dx}V_0^{(j)}(x)+(\lambda+\gamma_j(x))V_0^{(j)}(x)=0;\ 1\leq j\leq K \tag{9}$$

$$\frac{d}{dx}V_n^{(j)}(x)+(\lambda+\gamma_j(x))V_n^{(j)}(x)=\lambda V_{n-1}^{(j)}(x);\ \ n\geq 1, 1\leq j\leq K \tag{10}$$

$$\lambda Q=\int_0^\infty V_0^{(K)}(x)\gamma_K(x)dx \tag{11}$$

These set of equations are to be solved under the boundary conditions at x=0

$$P_0^{(1,1)}(0)=\lambda pQ+p\int_0^\infty V_1^{(K)}(x)\gamma_K(x)dx+p(1-r)\left\{\int_0^\infty P_1^{(1,2)}(x)\mu_{1,2}(x)dx+\int_0^\infty P_1^{(2,1)}(x)\mu_{2,1}(x)dx\right\}$$

$$+p\int_0^\infty P_1^{(2,2)}(x)\mu_{2,2}(x)dx \tag{12}$$

$$P_n^{(1,1)}(0)=p\int_0^\infty V_{n+1}^{(K)}(x)\gamma_K(x)dx+p(1-r)\left\{\int_0^\infty P_{n+1}^{(1,2)}(x)\mu_{1,2}(x)dx+\int_0^\infty P_{n+1}^{(2,1)}(x)\mu_{2,1}(x)dx\right\}$$

$$+p\int_0^\infty P_{n+1}^{(2,2)}(x)\mu_{2,2}(x)dx, n\geq 1 \tag{13}$$

$$P_n^{(1,2)}(0)=\int_0^\infty P_n^{(1,1)}(x)\mu_{1,1}(x)dx, n\geq 0 \tag{14}$$

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$$P_0^{(2,1)}(0) = \lambda(1-p)Q + (1-p)(1-r)\left\{\int_0^\infty P_1^{(1,2)}(x)\mu_{1,2}(x)dx + \int_0^\infty P_1^{(2,1)}(x)\mu_{2,1}(x)dx\right\}$$

$$+(1-p)\int_0^\infty P_1^{(2,2)}(x)\mu_{2,2}(x)dx + (1-p)\int_0^\infty V_1^{(K)}(x)\gamma_K(x)dx \tag{15}$$

$$P_n^{(2,1)}(0) = (1-p)\int_0^\infty V_{n+1}^{(K)}(x)\gamma_K(x)dx + (1-p)(1-r)\left\{\int_0^\infty P_{n+1}^{(1,2)}(x)\mu_{1,2}(x)dx + \int_0^\infty P_{n+1}^{(2,1)}(x)\mu_{2,1}(x)dx\right\}$$

$$+(1-p)\int_0^\infty P_{n+1}^{(2,2)}(x)\mu_{2,2}(x)dx, n \geq 1 \tag{16}$$

$$P_n^{(2,2)}(0) = r\int_0^\infty P_n^{(1,2)}(x)\mu_{1,2}(x)dx + r\int_0^\infty P_n^{(2,1)}(x)\mu_{2,1}(x)dx, n \geq 0 \tag{17}$$

$$V_0^{(1)}(0) = (1-r)\int_0^\infty P_0^{(1,2)}(x)\mu_{1,2}(x)dx + (1-r)\int_0^\infty P_0^{(2,1)}(x)\mu_{2,1}(x)dx + \int_0^\infty P_0^{(2,2)}(x)\mu_{2,2}(x)dx \tag{18}$$

$$V_n^{(1)} = 0,\ n \geq 1 \tag{19}$$

$$V_n^{(j)}(0) = \int_0^\infty V_n^{(j-1)}(x)\gamma_{j-1}(x)dx,\ 2 \leq j \leq K, n \geq 0 \tag{20}$$

with the normalization condition

$$Q + \int_0^\infty \left[\sum_{n=0}^\infty \sum_{j=1}^2 [P_n^{(1,j)}(x) + P_n^{(2,j)}(x)]dx + \sum_{n=0}^\infty \sum_{j=1}^K V_n^{(j)}(x)dx\right] = 1 \tag{21}$$

IV. THE ANALYSIS

The model defined in section3, has been solved using the probability generating functions defined in section 2.Multiplying equations (2), (4), (6), (8) and (10) by z^n, summing from $n=1$ to ∞ and then adding (1), (3), (5), (7) and (9), we get on simplification

$$\frac{\frac{d}{dx}P^{(i,l)}(x,z)}{P^{(i,l)}(x,z)} = -s - \mu_{i,l}(x) \tag{22}$$

$$\frac{\frac{d}{dx}V^{(j)}(x,z)}{V^{(j)}(x,z)} = -s - \gamma_j(x) \tag{23}$$

where $s = \lambda(1-z)$ and $i,l = 1,2;\ 1 \leq j \leq K.$

Integration of the equations (22) and (23) leads to

$$P^{(i,l)}(x,z) = C_{i,l}(1 - B_{i,l}(x))e^{-sx} \tag{24}$$

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$$V^{(j)}(x,z) = C_j(1-V_j(x))e^{-sx} \tag{25}$$

Taking $x=0$ in equations (24), (25), the constants $C_{i,l}$, C_j are obtained as

$$C_{i,l} = P^{(i,l)}(0,z) \tag{26}$$

$$C_j = V^{(j)}(0,z) \tag{27}$$

Using equations (26), (27) in (24), (25), we get

$$P^{(i,l)}(x,z) = P^{(i,l)}(0,z)(1-B_{i,l}(x))e^{-sx} \tag{28}$$

$$V^{(j)}(x,z) = V^{(j)}(0,z)(1-V_j(x))e^{-sx} \tag{29}$$

Multiplying equations (13), (16), (19) by z^n, summing from $n=1$ to ∞, adding (12), (15), (18) and using (11), (18), (28) and (29) with the corresponding equation, we get

$$zP^{(1,1)}(0,z) = \lambda p(z-1)Q + pV_K^*(s)V^{(K)}(0,z) + p(1-r)[B_{1,2}^*(s)P^{(1,2)}(0,z) + B_{2,1}^*(s)P^{(2,1)}(0,z)]$$
$$+ pB_{2,2}^*(s)P^{(2,2)}(0,z) - pV_0^{(1)}(0) \tag{30}$$

$$zP^{(2,1)}(0,z) = (1-p)[\lambda(z-1)Q + V_K^*(s)V^{(K)}(0,z) + (1-r)[B_{1,2}^*(s)P^{(1,2)}(0,z) + B_{2,1}^*(s)P^{(2,1)}(0,z)]]$$
$$+ (1-p)[B_{2,2}^*(s)P^{(2,2)}(0,z) - V_0^{(1)}(0)] \tag{31}$$

$$V^{(1)}(0,z) = V_0^{(1)}(0) \tag{32}$$

Now multiplying equation (14) by z^n, summing from $n=0$ to ∞ and using equation (28), we get

$$P^{(1,2)}(0,z) = B_{1,1}^*(s)P^{(1,1)}(0,z) \tag{33}$$

Performing similar operation on equations (17) and (20), we obtain

$$P^{(2,2)}(0,z) = rB_{1,2}^*(s)P^{(1,2)}(0,z) + rB_{2,1}^*(s)P^{(2,1)}(0,z) \tag{34}$$

$$V^{(j)}(0,z) = V_{j-1}^*(s)V^{(j-1)}(0,z),\ 2 \le j \le K \tag{35}$$

Putting $j = 2,3,\ldots,K$ in (35), using (32) and forward substitutions, we get

$$V^{(j)}(0,z) = \prod_{l=1}^{j-1} V_l^*(s)V_0^{(1)}(0),\ 2 \le j \le K \tag{36}$$

From equations (9) and (20), we get

$$V_0^{(j)}(x) = V_0^{(j)}(0)[1-V_j(x)]e^{-\lambda x},\ 1 \le j \le K \tag{37}$$

$$V_0^{(j)}(0) = V_0^{(j-1)}(0)V_{j-1}^*(\lambda),\ 2 \le j \le K \tag{38}$$

Putting $j = 2,3,\ldots,K$ in (38) and forward substitutions, we get

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$$V_0^{(j)}(0)=\prod_{l=1}^{j-1}V_l^*(\lambda)V_0^{(1)}(0),\ 2\le j\le K \tag{39}$$

Using equations (37) and (39) in (11), we get

$$\lambda Q=\prod_{l=1}^{K}V_l^*(\lambda)V_0^{(1)}(0) \tag{40}$$

Using equations (33), (34), (36) and (40) in (30), (31), we get

$$[z-pB_{1,1}^*(s)B_{1,2}^*(s)(1-r+rB_{2,2}^*(s))]P^{(1,1)}(0,z)=p\left[(z-1)\prod_{l=1}^{K}V_l^*(\lambda)+\prod_{l=1}^{K}V_l^*(s)-1\right]V_0^{(1)}(0)$$

$$+pB_{2,1}^*(s)(1-r+rB_{2,2}^*(s))P^{(2,1)}(0,z) \tag{41}$$

$$[z-(1-p)B_{2,1}^*(s)(1-r+rB_{2,2}^*(s))]P^{(2,1)}(0,z)=(1-p)\left[(z-1)\prod_{l=1}^{K}V_l^*(\lambda)+\prod_{l=1}^{K}V_l^*(s)-1\right]V_0^{(1)}(0)$$

$$+(1-p)B_{1,1}^*(s)B_{1,2}^*(s)(1-r+rB_{2,2}^*(s))P^{(1,1)}(0,z) \tag{42}$$

From equations (41) and (42), we get

$$P^{(1,1)}(0,z)=\frac{p\left[(z-1)\prod_{l=1}^{K}V_l^*(\lambda)+\prod_{l=1}^{K}V_l^*(s)-1\right]V_0^{(1)}(0)}{z-(1-r+rB_{2,2}^*(s))[pB_{1,1}^*(s)B_{1,2}^*(s)+(1-p)B_{2,1}^*(s)]} \tag{43}$$

$$P^{(2,1)}(0,z)=\frac{(1-p)\left[(z-1)\prod_{l=1}^{K}V_l^*(\lambda)+\prod_{l=1}^{K}V_l^*(s)-1\right]V_0^{(1)}(0)}{z-(1-r+rB_{2,2}^*(s))[pB_{1,1}^*(s)B_{1,2}^*(s)+(1-p)B_{2,1}^*(s)]} \tag{44}$$

Using equation (43) in (33), we get

$$P^{(1,2)}(0,z)=\frac{pB_{1,1}^*(s)\left[(z-1)\prod_{l=1}^{K}V_l^*(\lambda)+\prod_{l=1}^{K}V_l^*(s)-1\right]V_0^{(1)}(0)}{z-(1-r+rB_{2,2}^*(s))[pB_{1,1}^*(s)B_{1,2}^*(s)+(1-p)B_{2,1}^*(s)]} \tag{45}$$

Using equations (44) and (45) in (34), we get

$$P^{(2,2)}(0,z)=\frac{r[pB_{1,1}^*(s)B_{1,2}^*(s)+(1-p)B_{2,1}^*(s)]\left[(z-1)\prod_{l=1}^{K}V_l^*(\lambda)+\prod_{l=1}^{K}V_l^*(s)-1\right]V_0^{(1)}(0)}{z-(1-r+rB_{2,2}^*(s))[pB_{1,1}^*(s)B_{1,2}^*(s)+(1-p)B_{2,1}^*(s)]} \tag{46}$$

Integration of equations (28) and (29) by parts with respect to x and then using equations (43)- (46), we get

$$P^{(1,1)}(z)=\int_0^\infty P^{(1,1)}(x,z)dx$$

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$$=\frac{p(1-B_{1,1}^*(s))\left[(z-1)\prod_{l=1}^{K}V_l^*(\lambda)+\prod_{l=1}^{K}V_l^*(s)-1\right]V_0^{(1)}(0)}{s[z-(1-r+rB_{2,2}^*(s))(pB_{1,1}^*(s)B_{1,2}^*(s)+(1-p)B_{2,1}^*(s))]} \tag{47}$$

$$P^{(1,2)}(z)=\int_0^\infty P^{(1,2)}(x,z)dx$$

$$=\frac{pB_{1,1}^*(s)(1-B_{1,2}^*(s))\left[(z-1)\prod_{l=1}^{K}V_l^*(\lambda)+\prod_{l=1}^{K}V_l^*(s)-1\right]V_0^{(1)}(0)}{s[z-(1-r+rB_{2,2}^*(s))(pB_{1,1}^*(s)B_{1,2}^*(s)+(1-p)B_{2,1}^*(s))]} \tag{48}$$

$$P^{(2,1)}(z)=\int_0^\infty P^{(2,1)}(x,z)dx$$

$$=\frac{(1-p)(1-B_{2,1}^*(s))\left[(z-1)\prod_{l=1}^{K}V_l^*(\lambda)+\prod_{l=1}^{K}V_l^*(s)-1\right]V_0^{(1)}(0)}{s[z-(1-r+rB_{2,2}^*(s))(pB_{1,1}^*(s)B_{1,2}^*(s)+(1-p)B_{2,1}^*(s))]} \tag{49}$$

$$P^{(2,2)}(z)=\int_0^\infty P^{(2,2)}(x,z)dx$$

$$=\frac{r[pB_{1,1}^*(s)B_{1,2}^*(s)+(1-p)B_{2,1}^*(s)](1-B_{2,2}^*(s))\left[(z-1)\prod_{l=1}^{K}V_l^*(\lambda)+\prod_{l=1}^{K}V_l^*(s)-1\right]V_0^{(1)}(0)}{s[z-(1-r+rB_{2,2}^*(s))(pB_{1,1}^*(s)B_{1,2}^*(s)+(1-p)B_{2,1}^*(s))]} \tag{50}$$

$$V^{(1)}(z)=\int_0^\infty V^{(1)}(x,z)dx$$

$$=\frac{(1-V_1^*(s))V_0^{(1)}(0)}{s} \tag{51}$$

$$V^{(j)}(z)=\int_0^\infty V^{(j)}(x,z)dx,\ 2\le j\le K$$

$$=\frac{(1-V_j^*(s))\prod_{l=1}^{j-1}V_l^*(s)V_0^{(1)}(0)}{s} \tag{52}$$

Multiplying equation (21) by z^n and using the p.g.f, we get

$$Qz^n+\int_0^\infty\left[\sum_{j=1}^{2}[P^{(1,j)}(x,z)+P^{(2,j)}(x,z)]+\sum_{j=1}^{K}V^{(j)}(x,z)\right]dx=z^n$$

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$$Qz^n + \sum_{j=1}^{2}[P^{(1,j)}(z) + P^{(2,j)}(z)] + \sum_{j=1}^{K} V^{(j)}(z) = z^n$$

Putting $z = 1$ in above equation, we get

$$Q + \left[\sum_{j=1}^{2}[P^{(1,j)}(1) + P^{(2,j)}(1)] + \sum_{j=1}^{K} V^{(j)}(1)\right] = 1$$

Putting $z = 1$ in equations (47)-(52) and using above, we get

$$V_0^{(1)}(0) = \frac{\lambda[1 - \lambda p(E(B_{1,1}) + E(B_{1,2})) - \lambda(1-p)E(B_{2,1}) - \lambda r E(B_{2,2})]}{\left[\prod_{l=1}^{K} V_l^*(\lambda) + \lambda \sum_{j=1}^{K} E(V_j)\right]} \tag{53}$$

Using equation (53) in (40), we get

$$Q = \frac{\prod_{l=1}^{K} V_l^*(\lambda)[1 - \lambda p(E(B_{1,1}) + E(B_{1,2})) - \lambda(1-p)E(B_{2,1}) - \lambda r E(B_{2,2})]}{\left[\prod_{l=1}^{K} V_l^*(\lambda) + \lambda \sum_{j=1}^{K} E(V_j)\right]} \tag{54}$$

Equation (54) is called the probability that the server is idle and no customers in the system. Here $Q > 0$ guarantees the existence of the stability condition for the system takes the form $\rho < 1$ where

$$\rho = \lambda p(E(B_{1,1}) + E(B_{1,2})) + \lambda(1-p)E(B_{2,1}) + \lambda r E(B_{2,2}).$$

Equations (47)-(52) together with equation (53) are respectively, the probability generating functions of the number of customers in the queue when the server is, serving phase 1 service, serving phase 2 service, serving type 2 service, serving an optional service, the sever is on phase 1 vacation and the sever is on phase $j,\ 2 \le j \le K$ vacation respectively.

V. THE PERFORMANCE MEASURES

The next objective is to provide explicit expressions for some performance measures of the system. These performance measures are obtained using the properties of probability generating functions.

(i) The mean number of customers in the queue when the server provides type 1 service is

$$L_{q1} = \lim_{z \to 1} \frac{d}{dz}[P^{(1,1)}(z) + P^{(1,2)}(z)]$$

$$= \lim_{z \to 1} \frac{d}{dz}\left[\frac{p[1 - B_{1,1}^*(s)B_{1,2}^*(s)]\left[(z-1)\prod_{l=1}^{K} V_l^*(\lambda) + \prod_{l=1}^{K} V_l^*(s) - 1\right] V_0^{(1)}(0)}{s[z - (1 - r + rB_{2,2}^*(s))(pB_{1,1}^*(s)B_{1,2}^*(s) + (1-p)B_{2,1}^*(s))]}\right]$$

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$$L_{q1}=\frac{\lambda^3 p[E(B_{1,1})+E(B_{1,2})]\left[\sum_{j=1}^{K}E(V_j^2)+2\sum_{j=1}^{K-1}E(V_j)\sum_{l==j+1}^{K}E(V_l)\right]}{2\left[\prod_{l=1}^{K}V_l^*(\lambda)+\lambda\sum_{j=1}^{K}E(V_j)\right]}+\frac{\lambda^2 p}{2(1-\rho)}\{\lambda[E(B_{1,1})+E(B_{1,2})]$$
$$\times\left[rE(B_{2,2}^2)+(1-p)E(B_{2,1}^2)+2rE(B_{2,2})[pE(B_{1,1})+pE(B_{1,2})+(1-p)E(B_{2,1})]\right]+[1-\lambda rE(B_{2,2})$$
$$-\lambda(1-p)E(B_{2,1})][E(B_{1,1}^2)+E(B_{1,2}^2)+2E(B_{1,1})E(B_{1,2})]\} \qquad (55)$$

(ii) The mean number of customers in the queue when the server provides type 2 service is

$L_{q2}=\lim_{z\to 1}\frac{d}{dz}P^{(2,1)}(z)$ where $P^{(2,1)}(z)$ is given in equation (49).

$$L_{q2}=\frac{\lambda^3(1-p)E(B_{2,1})\left[\sum_{j=1}^{K}E(V_j^2)+2\sum_{j=1}^{K-1}E(V_j)\sum_{l==j+1}^{K}E(V_l)\right]}{2\left[\prod_{l=1}^{K}V_l^*(\lambda)+\lambda\sum_{j=1}^{K}E(V_j)\right]}+\frac{\lambda^2(1-p)}{2(1-\rho)}\{\lambda E(B_{2,1})\left[pE(B_{1,1}^2)+pE(B_{1,2}^2)\right.$$
$$\left.+rE(B_{2,2}^2)+2pE(B_{1,1})E(B_{1,2})+2rE(B_{2,2})[pE(B_{1,1})+pE(B_{1,2})+(1-p)E(B_{2,1})]\right]+E(B_{2,1}^2)$$
$$\times[1-\lambda p(E(B_{1,1})+E(B_{1,2}))-\lambda rE(B_{2,2})]\} \qquad (56)$$

(iii) The mean number of customers in the queue when the server provides optional service is

$L_{opt}=\lim_{z\to 1}\frac{d}{dz}P^{(2,2)}(z)$ where $P^{(2,2)}(z)$ is given in equation (50).

$$L_{opt}=\frac{\lambda^3 rE(B_{2,2})\left[\sum_{j=1}^{K}E(V_j^2)+2\sum_{j=1}^{K-1}E(V_j)\sum_{l==j+1}^{K}E(V_l)\right]}{2\left[\prod_{l=1}^{K}V_l^*(\lambda)+\lambda\sum_{j=1}^{K}E(V_j)\right]}+\frac{\lambda^2 r}{2(1-\rho)}\{\lambda E(B_{2,2})\left[pE(B_{1,1}^2)+pE(B_{1,2}^2)\right.$$
$$\left.+(1-p)E(B_{2,1}^2)+2pE(B_{1,1})E(B_{1,2})\right]+[1-\lambda p(E(B_{1,1})+E(B_{1,2}))-\lambda(1-p)E(B_{2,1})][E(B_{2,2}^2)$$
$$+2E(B_{2,2})[pE(B_{1,1})+pE(B_{1,2})+(1-p)E(B_{2,1})]]\} \qquad (57)$$

(iv) The mean number of customers in the queue when the server is on type $j,\ 1\le j\le K$ vacation is

$L_{qv}^{(1)}=\lim_{z\to 1}\frac{d}{dz}V^{(1)}(z)$ where $V^{(1)}(z)$ is given in equation (51).

$$=\frac{\lambda^2 E(V_1^2)(1-\rho)}{2\left[\prod_{l=1}^{K}V_l^*(\lambda)+\lambda\sum_{j=1}^{K}E(V_j)\right]}$$

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$L_{qv}^{(j)} = \lim_{z\to 1}\frac{d}{dz}V^{(j)}(z),\ 2\le j\le K$ where $V^{(j)}(z)$ is given in equation (52).

$$= \frac{\lambda\left[2E(V_j)\sum_{l=1}^{j-1}E(V_l)+\lambda E(V_j^2)\right](1-\rho)}{2\left[\prod_{l=1}^{K}V_l^*(\lambda)+\lambda\sum_{j=1}^{K}E(V_j)\right]}$$

(v) The mean waiting time when the server provides the type 1 service

$$W_{q1} = \frac{L_{q1}}{\lambda}$$

where L_{q1} is given in equation (55).

(vi) The mean waiting time when the server provides the type 2 service

$$W_{q2} = \frac{L_{q2}}{\lambda}$$

where L_{q2} is given in equation (56).

(vii) The mean waiting time when the server provides the optional service

$$W_{opt} = \frac{L_{opt}}{\lambda}$$

where L_{opt} is given in equation (57).

(viii) The probability that the server is busy

- with type 1 service is $P_b^{(1)} = \sum_{j=1}^{2}P^{(1,j)}(1) = \lambda p(E(B_{1,1})+E(B_{1,2}))$
- with type 2 service is $P_b^{(2,1)} = \lambda(1-p)E(B_{2,1})$
- with an optional service is $P_b^{(2,2)} = \lambda rE(B_{2,2})$

(ix) The probability that the server is on type j vacation is

$$P_v^{(j)} = \frac{\lambda E(V_j)[1-\lambda p(E(B_{1,1})+E(B_{1,2}))-\lambda(1-p)E(B_{2,1})-\lambda rE(B_{2,2})]}{\left[\prod_{l=1}^{K}V_l^*(\lambda)+\lambda\sum_{j=1}^{K}E(V_j)\right]},\ 1\le j\le K$$

(x) The utilization factor is

$$\rho = \lambda p(E(B_{1,1})+E(B_{1,2}))+\lambda(1-p)E(B_{2,1})+\lambda rE(B_{2,2})$$

VI. PARTICULAR MODEL

In this section, some particular models are calculated using the results derived in the sections 4 and 5.

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Model 1: The service times and vacation times distributions follows a negative exponential distribution with parameters $\mu_{1,1}$ for phase 1, $\mu_{1,2}$ for phase 2, $\mu_{2,1}$ for type 2, $\mu_{2,2}$ for an optional service and θ_j, $(1 \le j \le K)$ for vacation.

$$\rho = \frac{\lambda}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}}\left\{p\mu_{2,1}\mu_{2,2}(\mu_{1,1}+\mu_{1,2})+\mu_{1,1}\mu_{1,2}[(1-p)\mu_{2,2}+r\mu_{2,1}]\right\}$$

$$Q = \frac{D_1\prod_{l=1}^{K}\frac{\theta_l}{\lambda+\theta_l}}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}\left[\prod_{l=1}^{K}\frac{\theta_l}{\lambda+\theta_l}+\lambda\sum_{j=1}^{K}\frac{1}{\theta_j}\right]}$$

$$P_b^{(1)} = \frac{\lambda p(\mu_{1,1}+\mu_{1,2})}{\mu_{1,1}\mu_{1,2}}$$

$$P_b^{(2,1)} = \frac{\lambda(1-p)}{\mu_{2,1}}$$

$$P_b^{(2,1)} = \frac{\lambda r}{\mu_{2,2}}$$

$$P_v^{(j)} = \frac{\lambda D_1}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}\theta_j\left[\prod_{l=1}^{K}\frac{\theta_l}{\lambda+\theta_l}+\lambda\sum_{j=1}^{K}\frac{1}{\theta_j}\right]},\ 1 \le j \le K$$

$$L_{q1} = \frac{\lambda^3 p(\mu_{1,1}+\mu_{1,2})\left[\sum_{j=1}^{K}\frac{1}{\theta_j^2}+\sum_{j=1}^{K-1}\theta_j\sum_{l=j+1}^{K}\frac{1}{\theta_l}\right]}{\mu_{1,1}\mu_{1,2}\left[\prod_{l=1}^{K}\frac{\theta_l}{\lambda+\theta_l}+\lambda\sum_{j=1}^{K}\frac{1}{\theta_j}\right]}+\frac{\lambda^2 p}{D_1\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}}\left\{\lambda[\mu_{1,1}+\mu_{1,2}]\left[r\mu_{2,1}^2[\mu_{1,1}\mu_{1,2}\right.\right.$$
$$\left.+p\mu_{2,2}(\mu_{1,1}+\mu_{1,2})]+\mu_{1,1}\mu_{1,2}[(1-p)\mu_{2,2}(\mu_{2,2}+r\mu_{2,1})+r\mu_{2,1}^2]\right]+\mu_{2,1}\mu_{2,2}[\mu_{2,1}\mu_{2,2}-\lambda r\mu_{2,1}$$
$$\left.-\lambda(1-p)\mu_{2,2}][\mu_{1,1}^2+\mu_{1,2}^2+\mu_{1,1}\mu_{1,2}]\right\}$$

$$L_{q2} = \frac{\lambda^3 (1-p)\left[\sum_{j=1}^{K}\frac{1}{\theta_j^2}+\sum_{j=1}^{K-1}\theta_j\sum_{l=j+1}^{K}\frac{1}{\theta_l}\right]}{\mu_{2,1}\left[\prod_{l=1}^{K}\frac{\theta_l}{\lambda+\theta_l}+\lambda\sum_{j=1}^{K}\frac{1}{\theta_j}\right]}+\frac{\lambda^2(1-p)}{D_1\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}}\left\{\mu_{1,1}\mu_{1,2}\left[\mu_{1,1}\mu_{1,2}[\lambda r(\mu_{2,1}-p\mu_{2,2})\right.\right.$$
$$\left.\left.+\mu_{2,2}^2]+\lambda p\mu_{2,2}(\mu_{1,1}+\mu_{1,2})(r\mu_{2,1}-\mu_{2,2})]\right]+\lambda p\mu_{2,2}^2\mu_{2,1}[\mu_{1,1}^2+\mu_{1,2}^2+\mu_{1,1}\mu_{1,2}]\right\}$$

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$$L_{opt} = \frac{\lambda^3 r\left[\sum_{j=1}^{K}\frac{1}{\theta_j^2} + \sum_{j=1}^{K-1}\theta_j \sum_{l=j+1}^{K}\frac{1}{\theta_l}\right]}{\mu_{2,2}\left[\prod_{l=1}^{K}\frac{\theta_l}{\lambda+\theta_l} + \lambda\sum_{j=1}^{K}\frac{1}{\theta_j}\right]} + \frac{\lambda^2 r}{D_1\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}}\{\lambda\mu_{2,2}[p\mu_{2,1}^2[\mu_{1,1}^2+\mu_{1,2}^2+\mu_{1,1}\mu_{1,2})+(1-p)$$

$$\times\mu_{1,1}^2\mu_{1,2}^2]]+[\mu_{1,1}\mu_{1,2}(\mu_{2,1}-\lambda(1-p))-\lambda p\mu_{2,1}(\mu_{1,1}+\mu_{1,2})][\mu_{1,1}\mu_{1,2}(\mu_{2,1}+(1-p)\mu_{2,2})+p\mu_{2,1}$$

$$\times\mu_{1,2}(\mu_{1,1}+\mu_{1,2})]\}$$

$$L_{qv}^{(1)} = \frac{\lambda^2 D_1}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}\theta^2\left[\prod_{l=1}^{K}\frac{\theta_l}{\lambda+\theta_l} + \lambda\sum_{j=1}^{K}\frac{1}{\theta_j}\right]}$$

$$L_{qv}^{(j)} = \frac{\lambda D_1\left[\frac{1}{\theta_j}\sum_{l=1}^{j-1}\frac{1}{\theta_l} + \frac{\lambda}{\theta_j^2}\right]}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2}\left[\prod_{l=1}^{K}\frac{\theta_l}{\lambda+\theta_l} + \lambda\sum_{j=1}^{K}\frac{1}{\theta_j}\right]};2\le j\le K$$

where

$$D_1 = \mu_{1,1}\mu_{1,2}\mu_{2,1}\mu_{2,2} - \lambda p\mu_{2,1}\mu_{2,2}(\mu_{1,1}+\mu_{1,2}) - \lambda\mu_{1,1}\mu_{1,2}[(1-p)\mu_{2,2}+r\mu_{2,1}]$$

Model 2: $M/G/1$ ***Queue with Two Types of Service and with Single Vacation.***

The service times and vacation time follows a general distribution.

$$\rho = \lambda p(E(B_{1,1})+E(B_{1,2}))+\lambda(1-p)E(B_{2,1})$$

$$Q = \frac{V^*(\lambda)(1-\rho)}{V^*(\lambda)+\lambda E(V)}$$

$$P_b^{(1)} = \lambda p(E(B_{1,1})+E(B_{1,2}))$$

$$P_b^{(2,1)} = \lambda(1-p)E(B_{2,1})$$

$$P_v = \frac{\lambda E(V)(1-\rho)}{V^*(\lambda)+\lambda E(V)}$$

$$L_{q1} = \frac{\lambda^3 pE(V^2)[E(B_{1,1})+E(B_{1,2})]}{2[V^*(\lambda)+\lambda E(V)]} + \frac{\lambda^2 p}{2(1-\rho)}\{\lambda(1-p)E(B_{2,1}^2)(E(B_{1,1})+E(B_{1,2}))+[1-\lambda(1-p)$$

$$\times E(B_{2,1})][E(B_{1,1}^2)+E(B_{1,2}^2)+2E(B_{1,1})E(B_{1,2})]\}$$

$$L_{q2} = \frac{\lambda^3(1-p)E(B_{2,1})E(V^2)}{2[V^*(\lambda)+\lambda E(V)]} + \frac{\lambda^2(1-p)}{2(1-\rho)}\{\lambda E(B_{2,1})[pE(B_{1,1}^2)+pE(B_{1,2}^2)+2pE(B_{1,1})E(B_{1,2})]$$

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$$+E(B_{2,1}^2)[1-\lambda p(E(B_{1,1})+E(B_{1,2}))]\}$$

$$L_{qv}=\frac{\lambda^2 E(V^2)(1-\rho)}{2[V^*(\lambda)+\lambda E(V)]}$$

Model 3: $M/M/1$ ***Queue with Two Types of Service and with Single Vacation.***

In the above model, the service times and vacation time distributions follows a negative exponential distribution with parameters $\mu_{1,1}$ for phase 1, $\mu_{1,2}$ for phase 2, $\mu_{2,1}$ for type 2 and θ for vacation.

$$\rho=\frac{1}{\mu_{1,1}\mu_{1,2}\mu_{2,1}}\{p\mu_{2,1}(\mu_{1,1}+\mu_{1,2})+(1-p)\mu_{1,1}\mu_{1,2}\}$$

$$Q=\frac{\theta^2 D_2}{\mu_{1,1}\mu_{1,2}\mu_{2,1}[\theta^2+\lambda(\theta+\lambda)]}$$

$$P_b^{(1)}=\frac{\lambda p(\mu_{1,1}+\mu_{1,2})}{\mu_{1,1}\mu_{1,2}}$$

$$P_b^{(2,1)}=\frac{\lambda(1-p)}{\mu_{2,1}}$$

$$P_v=\frac{\lambda(\theta+\lambda)D_2}{\mu_{1,1}\mu_{1,2}\mu_{2,1}[\theta^2+\lambda(\theta+\lambda)]}$$

$$L_{q1}=\frac{\lambda^3 p(\mu_{1,1}+\mu_{1,2})(\theta+\lambda)}{\mu_{1,1}\mu_{1,2}\theta[\theta^2+\lambda(\theta+\lambda)]}+\frac{\lambda^2 p}{\mu_{1,1}\mu_{1,2}\mu_{2,1}D_2}\{\lambda(1-p)\mu_{1,1}\mu_{1,2}(\mu_{1,1}+\mu_{1,2})+\mu_{2,1}[\mu_{2,1}-\lambda(1-p)]$$

$$\times[\mu_{1,1}^2+\mu_{1,2}^2+\mu_{1,1}\mu_{1,2}]\}$$

$$L_{q2}=\frac{\lambda^3(1-p)(\theta+\lambda)}{\mu_{2,1}\theta[\theta^2+\lambda(\theta+\lambda)]}+\frac{\lambda^2(1-p)}{\mu_{1,1}\mu_{1,2}\mu_{2,1}D_2}\{\mu_{1,1}\mu_{1,2}[\mu_{1,1}\mu_{1,2}-\lambda p(\mu_{1,1}+\mu_{1,2})]+\lambda p\mu_{2,1}$$

$$\times[\mu_{1,1}^2+\mu_{1,2}^2+\mu_{1,1}\mu_{1,2}]\}$$

$$L_{qv}=\frac{\lambda^2(\theta+\lambda)D_2}{\mu_{1,1}\mu_{1,2}\mu_{2,1}\theta[\theta^2+\lambda(\theta+\lambda)]}$$

where

$$D_2=\mu_{1,1}\mu_{1,2}\mu_{2,1}-\lambda p\mu_{2,1}(\mu_{1,1}+\mu_{1,2})-\lambda(1-p)\mu_{1,1}\mu_{1,2}$$

VII. THE NUMERICAL STUDY

This section deals with some illustrations corresponding to the models discussed in section 6. Two numerical models have been presented. First model is a single server Markovian queue with K phases of (case I) vacation, two types of service and an optional service. The second model is a single server Markovian queue with two types of

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services and with a single vacation (case II). The two cases are completely analyzed by finding the performance measures, namely, mean numbers of customer at various server state, mean waiting time, the idle probability, the busy probability, the probability that the server is on vacation. The results are presented in figures and tables.

Case I: A single server Markovian queue with K phase of vacation, two types of services and an optional service (Model I).

For the analysis the parameter $p = 0.3, r = 0.4, \theta = 1.3, \mu_{11} = 1.7, \mu_{12} = 2.0, \mu_{21} = 1.8, \mu_{22} = 2.5,$ have been fixed and the parameters $k = 2(2)10$, $\lambda = 0.1(0.1)1.0$ have been varied. The service times and vacation times follow negative exponential distributions. In figures 1 to 5, The graphs represent the curves of L_{q_1}, L_{q_2} of L_{opt} as a function of arrival rate λ. The curves for $k = 2,4,6,8,10$ are presented in figure 1 to 5 respectively. From the figures it is clear that all the curves are increasing curves with respect to arrival rateλ. The figures 6 to 10, The graphs represent the curves of $L_{v_1}, L_{v_2}, \dots L_{v_k}$ for $k = 2, k = 4, k = 6, k = 8, k = 10$. In these figures all the curves are convex curves with respect to arrival rate λ.The figures 11 to 15, The curves of $w_{q_1}, w_{q_2}, w_{opt}$ are drawn with respect to arrival rate. The graphs represent the curves of $w_{q_1}, w_{q_2}, w_{opt}$ for $k = 2,4,6,8,10.$ All the curves are increasing curves with respect to arrival rate λ.

The idle probability for $k = 2, k = 4, k = 6, k = 8,$ and $k = 10$ are presented in table 1. From the table, it is clear that the probabilities increases as arrival rate decreases. In table 2, for variance values of$k$, the utilization factor ρ, the busy probabilities $p_b^{(1)}, p_b^{(2,1)}, p_b^{(2,2)}$ are presented. From the table, the utilization factor increases for increasing value of λ. The same impact has been experienced in the case of the probabilities. Table 3, presents the probabilities of $p_v^{(1)}, p_v^{(2)}, \dots, p_v^{(k)}$for $k = 2,4,6,8,$ and 10.

Case II: A single server Markovian queue with two type of services and with single vacation (Model 3).

The parameter $p = 0.3, \theta = 1.3, \mu_{11} = 1.7, \mu_{12} = 2.0, \mu_{21} = 1.8, \mu_{22} = 2.5,$ have been fixed and the parameter $\lambda = 0.1(0.1)1.0$ have been varied. The curves of L_{q_1}, L_{q_2} of $L_{q_v}$are presented in figure 16 and the curves ofw_{q_1}, w_{q_2}, are represents in figure 17. The probabilities of $p_v^{(1)}, p_v^{(1,2)}, p_b^{(2,1)}, p_v, Q$ and utilization factor ρ has been presented in table 4.

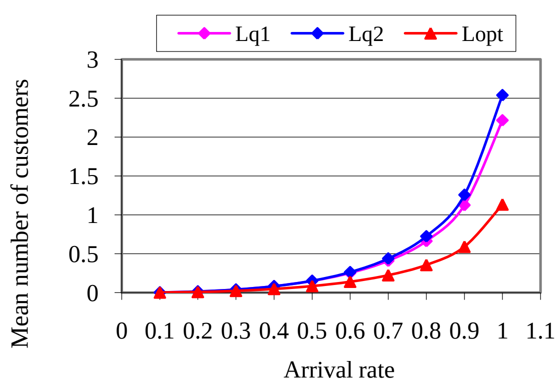

Figure 1: Arrival Rate Versus Mean Number of Customers for K=2

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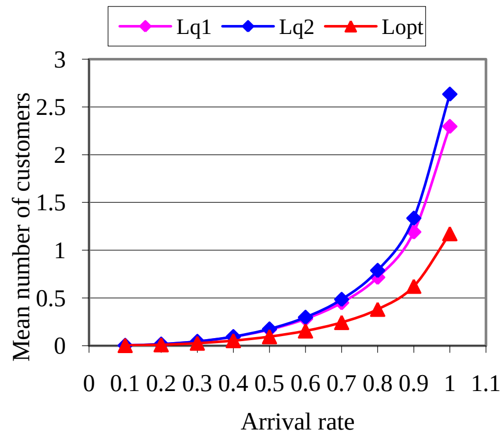

Figure 2: Arrival Rate Versus Mean Number Customers for K=4

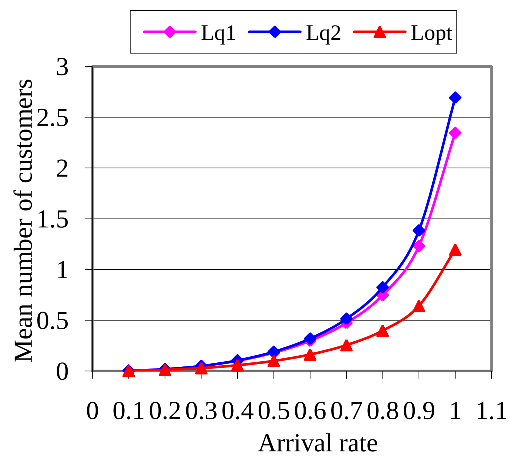

Figure 3: Arrival Rate Versus Mean Number of Customers for K=6

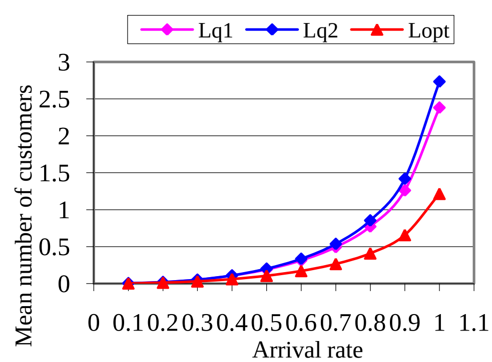

Figure 4: Arrival Rate Versus Mean Number of Customers for K=8

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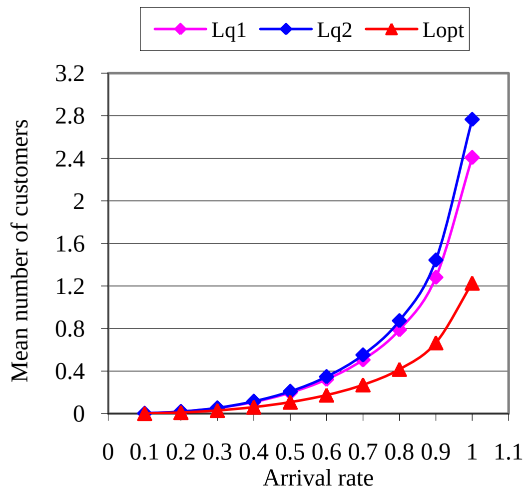

Figure 5: Arrival Rate Versus Mean Number of Customers for K=10

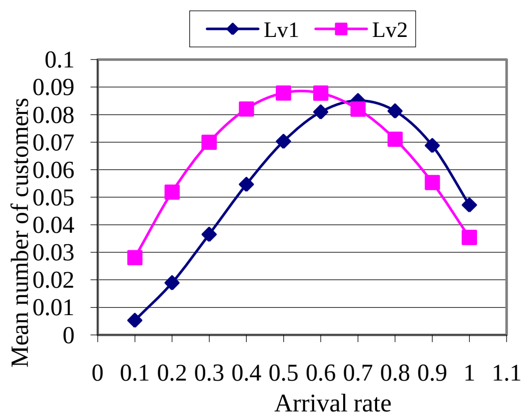

Figure 6: Arrival Rate Versus Mean Number of Customers for K=2

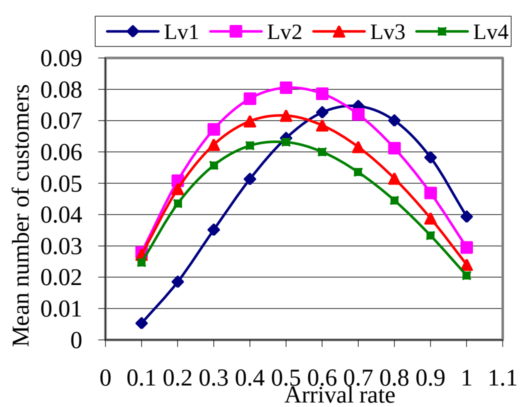

Figure 7: Arrival Rate Versus Mean Number of Customers for K=4

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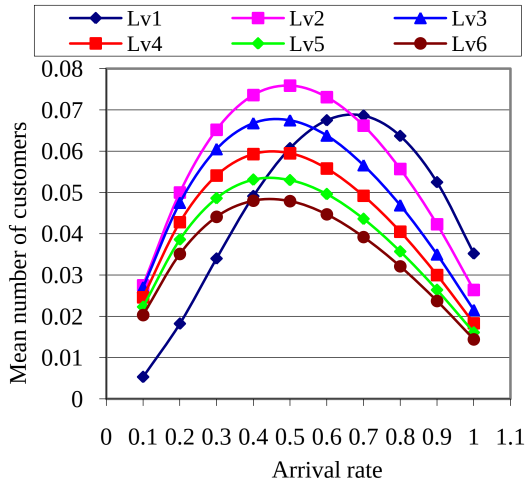

Figure 8: Arrival Rate Versus Mean Number of Customers for K=6

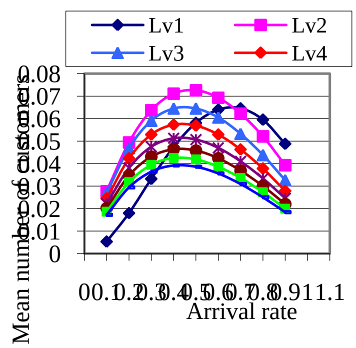

Figure 9: Arrival Rate Versus Mean Number of Customers for K=8

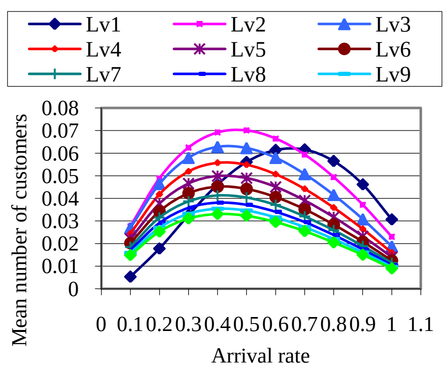

Figure 10: Arrival Rate Versus Mean Number of Customers for K=10

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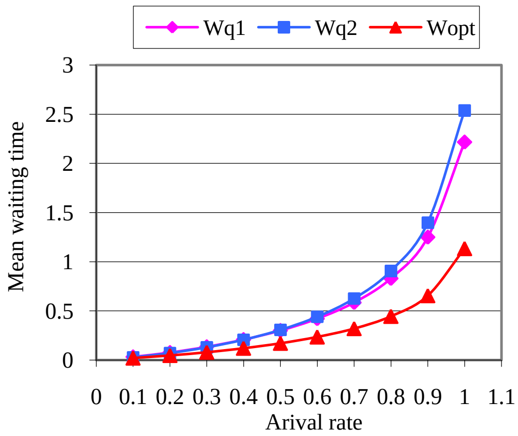

Figure 11: Arrival Rate Versus Mean Waiting Time for K=2

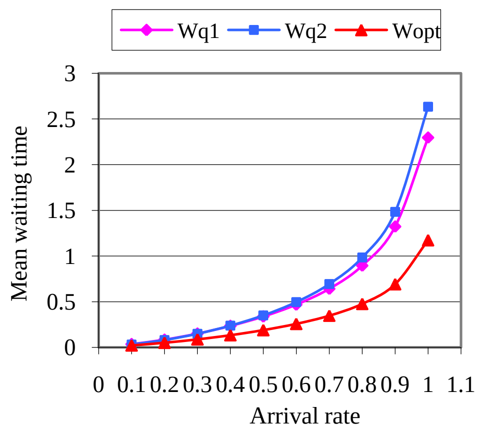

Figure 12: Arrival Rate Versus Mean Wating Time for K=4

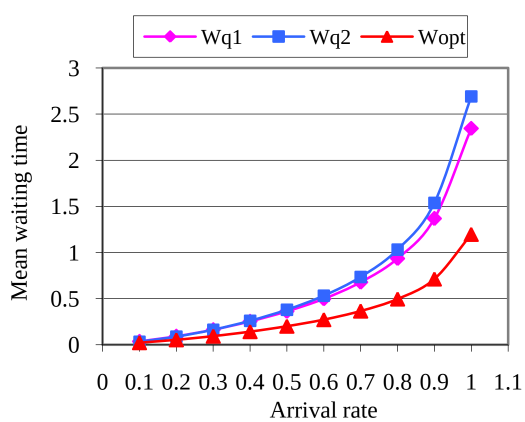

Figure 13: Arrival Rate Versus Mean Wating Time for K=6

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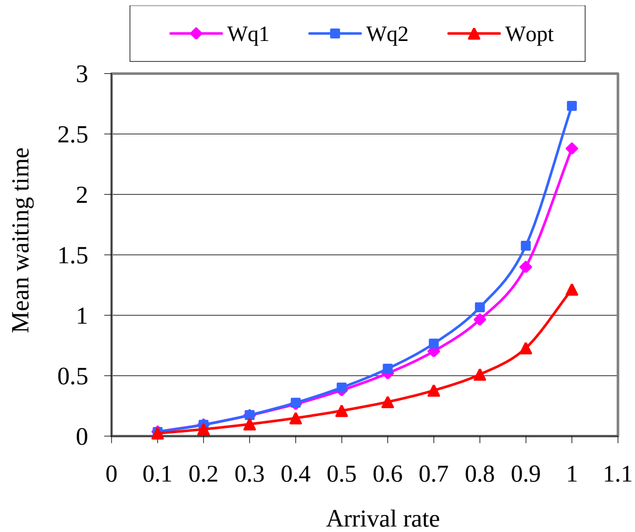

Figure 14: Arrival Rate Versus Mean Wating Time for K=8

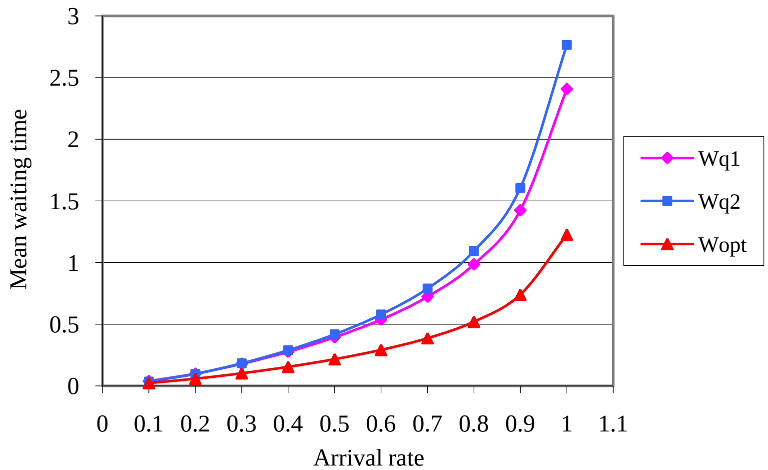

Figure 15: Arrival Rate Versus Mean Wating Time for K=10

Table 1: The Idle Probability (Q)

λ	$K = 2$	$K = 4$	$K = 6$	$K = 8$	$K = 10$
0.1	0.8082	0.7685	0.0739	0.7262	0.7123
0.2	0.6411	0.5749	0.5355	0.5078	0.4866
0.3	0.4999	0.4209	0.3761	0.3458	0.3234
0.4	0.3831	0.3025	0.2592	0.2310	0.2108
0.5	0.2880	0.2133	0.1755	0.1518	0.1353
0.6	0.2115	0.1472	0.1164	0.0979	0.0853
0.7	0.1503	0.0985	0.0750	0.0614	0.0524
0.8	0.1016	0.0629	0.0462	0.0369	0.0309
0.9	0.0630	0.0369	0.0262	0.0204	0.0168
1.0	0.0326	0.0181	0.0125	0.0095	0.0076

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Table 2: The Probabilities

λ	ρ	$P_b^{(1)}$	$P_b^{(2,1)}$	$P_b^{(2,2)}$
0.1	0.0875	0.0326	0.0389	0.0160
0.2	0.1751	0.0653	0.0778	0.0320
0.3	0.2626	0.0979	0.1167	0.0480
0.4	0.3501	0.1306	0.1556	0.0640
0.5	0.4377	0.1632	0.1944	0.0800
0.6	0.5252	0.1959	0.2333	0.0960
0.7	0.6128	0.2285	0.2722	0.1120
0.8	0.7003	0.2612	0.3111	0.1280
0.9	0.7878	0.2938	0.3500	0.1440
1.0	0.8754	0.3265	0.3889	0.1600

Table 3: The Probabilities

ds	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
$K=2$										
$P_v^{(1)}$	0.0695	0.1226	0.1584	0.1779	0.1829	0.1756	0.1580	0.1321	0.0994	0.0614
$P_v^{(2)}$	0.0348	0.0613	0.0792	0.0889	0.0914	0.0878	0.0790	0.0660	0.0497	0.0307
$K=4$										
$P_v^{(1)}$	0.0691	0.1200	0.1519	0.1667	0.1675	0.1573	0.1386	0.1137	0.0841	0.0511
$P_v^{(2)}$	0.0346	0.0600	0.0759	0.0834	0.0838	0.0786	0.0693	0.0568	0.0421	0.0256
$P_v^{(3)}$	0.0230	0.0400	0.0506	0.0556	0.0558	0.0524	0.0462	0.0379	0.0280	0.0170
$P_v^{(4)}$	0.0173	0.0300	0.0380	0.0417	0.0419	0.0393	0.0346	0.0284	0.0210	0.1028
$K=6$										
$P_v^{(1)}$	0.0688	0.1181	0.1475	0.1595	0.1579	0.1463	0.1274	0.1035	0.0759	0.0458
$P_v^{(2)}$	0.0344	0.0591	0.0737	0.0797	0.0789	0.0731	0.0637	0.0517	0.0379	0.0229
$P_v^{(3)}$	0.0229	0.0394	0.0492	0.0532	0.0526	0.0488	0.0425	0.0345	0.0253	0.0153
$P_v^{(4)}$	0.0172	0.0295	0.0369	0.0399	0.0395	0.0366	0.0319	0.0259	0.0190	0.0114
$P_v^{(5)}$	0.0138	0.0236	0.0295	0.0319	0.0316	0.0293	0.0255	0.0207	0.0152	0.0092
$P_v^{(6)}$	0.0115	0.0197	0.0246	0.0266	0.0263	0.0244	0.0212	0.0172	0.0126	0.0076
$K=8$										
$P_v^{(1)}$	0.0685	0.1167	0.1441	0.1541	0.1510	0.1387	0.1199	0.0967	0.0705	0.0424
$P_v^{(2)}$	0.0343	0.0583	0.0720	0.0770	0.0755	0.0693	0.0599	0.0484	0.0353	0.0212
$P_v^{(3)}$	0.0228	0.0389	0.0480	0.0514	0.0503	0.0462	0.0400	0.0322	0.0235	0.0141
$P_v^{(4)}$	0.0171	0.0292	0.0360	0.0385	0.0378	0.0347	0.0300	0.0242	0.0176	0.0106
$P_v^{(5)}$	0.0137	0.0233	0.0288	0.0308	0.0302	0.0277	0.0240	0.0193	0.0141	0.0085
$P_v^{(6)}$	0.0114	0.0194	0.0240	0.0257	0.0252	0.0231	0.0200	0.0161	0.0118	0.0071
$P_v^{(7)}$	0.0098	0.0167	0.0206	0.0220	0.0216	0.0198	0.0171	0.0138	0.0101	0.0061
$P_v^{(8)}$	0.0086	0.0146	0.0180	0.0193	0.0189	0.0173	0.0150	0.0121	0.0088	0.0053
$K=10$										

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$P_v^{(1)}$	0.0683	0.1155	0.1414	0.1499	0.1458	0.1330	0.1143	0.0918	0.0667	0.0399
$P_v^{(2)}$	0.0342	0.0578	0.0707	0.0749	0.0729	0.0665	0.0572	0.0459	0.0334	0.0200
$P_v^{(3)}$	0.0228	0.0385	0.0471	0.0500	0.0486	0.0443	0.0381	0.0306	0.0222	0.0133
$P_v^{(4)}$	0.0171	0.0289	0.0353	0.0375	0.0364	0.0332	0.0286	0.0229	0.0167	0.0100
$P_v^{(5)}$	0.0137	0.0231	0.0283	0.0300	0.0292	0.0266	0.0229	0.0184	0.0133	0.0080
$P_v^{(6)}$	0.0114	0.0193	0.0236	0.0250	0.0243	0.0222	0.0191	0.0153	0.0111	0.0067
$P_v^{(7)}$	0.0098	0.0165	0.0202	0.0214	0.0208	0.0190	0.0163	0.0131	0.0095	0.0057
$P_v^{(8)}$	0.0085	0.0144	0.0177	0.0187	0.0182	0.0166	0.0143	0.0115	0.0083	0.0050
$P_v^{(9)}$	0.0076	0.0128	0.0157	0.0167	0.0162	0.0148	0.0127	0.0102	0.0074	0.0044
$P_v^{(10)}$	0.0068	0.0116	0.0141	0.0150	0.0146	0.0133	0.0114	0.0092	0.0067	0.0040

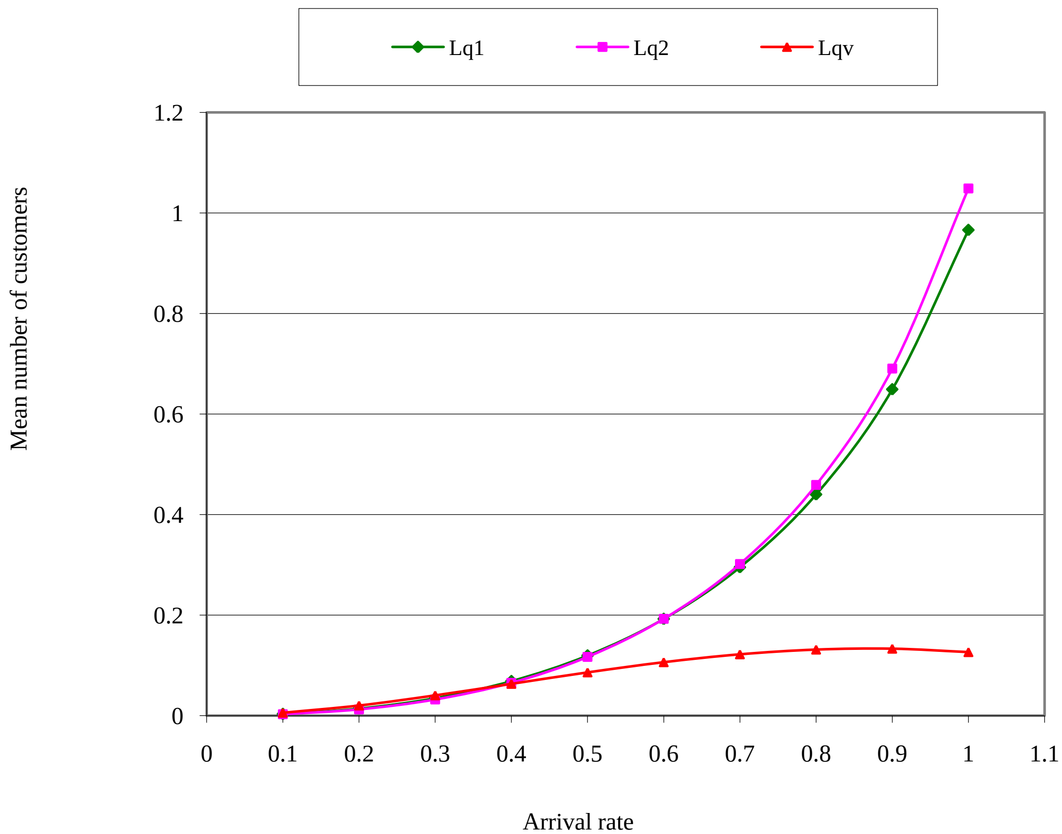

Figure 16: Arrival Rate Versus Mean Number of Customers

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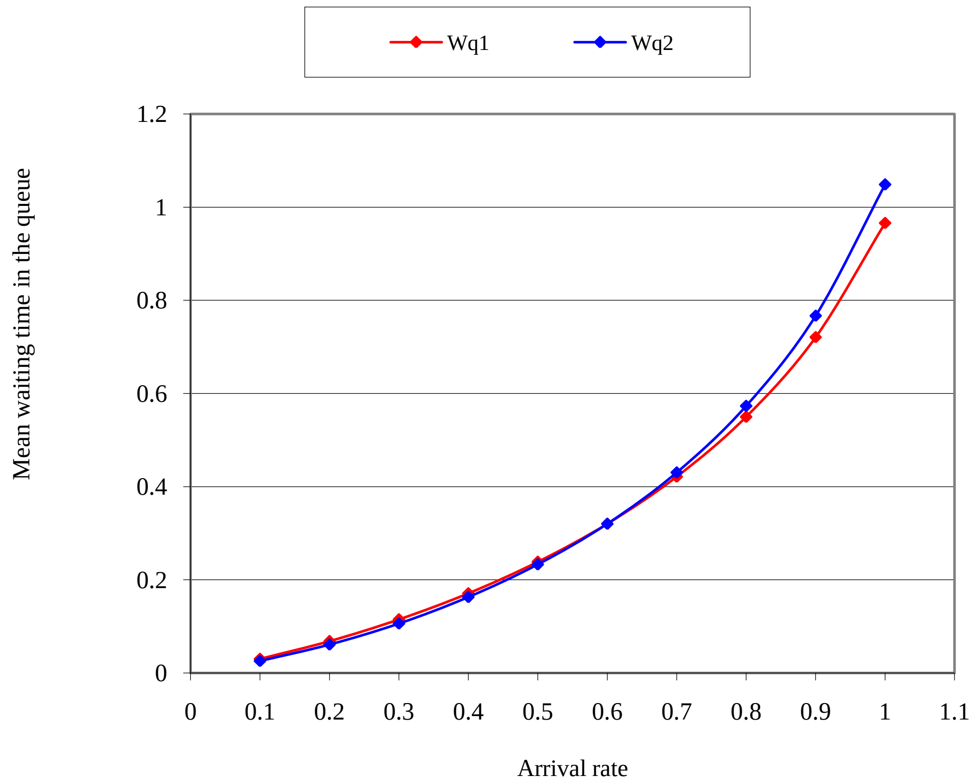

Figure 17: Arrival Rate Versus Mean Waiting Time in the Queue

Table 4: The Probabilities

λ	ρ	$P_b^{(1)}$	$P_b^{(2,1)}$	P_v	Q
0.1	0.0715	0.0326	0.0389	0.0710	0.8574
0.2	0.1431	0.0653	0.0778	0.1292	0.7277
0.3	0.2146	0.0979	0.1167	0.1737	0.6117
0.4	0.2861	0.1306	0.1556	0.2048	0.5090
0.5	0.3577	0.1632	0.1944	0.2232	0.4191
0.6	0.4292	0.1959	0.2333	0.2299	0.3409
0.7	0.5008	0.2285	0.2722	0.2262	0.2731
0.8	0.5723	0.2612	0.3111	0.2132	0.2145
0.9	0.6438	0.2938	0.3500	0.1922	0.1640
1.0	0.7154	0.3265	0.3889	0.1641	0.1206

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VIII. CONCLUSION

In this, we consider a single server queue with two types of services and with vacation has been considered. The type 1 service is a phase type service with two service phases. Both the service time distributions are generally distributed. The type 2 service has only one phase of service. In addition the server also provides an optional service. This service time distribution is also general. At each time the system becomes empty, the server takes K phases of vacation and the vacation time distribution of each phase is general. This model has been completely analysed in steady state. This model can be generalized by including more servers, but the analysis is not easy.

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