# TRACKING CONTROL OF QUANTUM VON NEUMANN ENTROPY

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***Abstract-*** *This paper provides the accurate tracking control of the von-Neumann entropy of quantum systems. The closed form of the accurate time derivative of the entropy is presented. Based on this, controller design method for time-dependent decay rate is provided, which can directly drive the entropy to track a desired trajectory. Simulations are done on two-level quantum system.*

***Indexterms-****Quantum control, Von Neumann entropy, Quantum information.*

## I. INTRODUCTION

Von Neumann entropy is very important in measure of entanglement, quantum systempurification, quantum cooling and information flow control. In the 2012 “Quantum Characterization, Verification and Validation Workshop” in Bethesda, a group of scientists discussed about some typical questions like how to build a quantum device. The consensus at this workshop is that our common goal should be to somehow master a quantum system’s entropy, thereby enabling smooth sailing towards our final destination of full-scale quantum computers.

A recent paper on twenty open problems in quantum control [1] mentioned that the difference between the maximum and minimum eigenvalues of the interaction Hamiltonian bounds the rate at which we can extract information from the system, which in turn bounds the rate at which we can extract entropy and reduce the effects of noise. However, it is not clear whether there is a way to compare the eigenvalue bound for unitary interactions to the damping rate of an irreversible coupling. Here the von Neumann entropy control is essential because the difference between the maximum and minimum eigenvalues is directly correlated to the entropy.

In quantum cooling [2-5], both preparing quantum systems in pure states and ground-state cooling require that all the entropy is extracted from the system. Pure states are not necessarily cooler than mixed states, for example, pure states with an equal superposition of all eigenstates correspond to infinite-temperature states. To achieve cooling, in general, we need to prepare states that have low entropy.

There have been many studies about how to calculate the time evolution of von Neumann entropy [6-11]. The accurate evolution of entropy should be unique, while how to calculate the accurate solution is still unsolved. This paper provides the accurate tracking control of von Neumann entropy, which can drive the entropy to track a pre-specified trajectory.

This paper is organized as follows. Section 2 gives the problem formulation and the closed form of the accurate time derivative of von Neumann entropy. Section 3 provides the controller design method for time-dependent decay rate. Concluding remarks are given in the Conclusion Section.

## II. PROBLEM FORMULATION

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In the frame of open quantum system, it's important to study how to drive the von-Neumann entropy from the system to the environment. The von-Neumann entropy of a quantum system can be defined as

$$S(t) = -\mathrm{Tr}_{\mathcal{S}}[\rho^{\mathcal{S}}(t)\ln \rho^{\mathcal{S}}(t)], \qquad (1)$$

where $\rho^{\mathcal{S}}(t)$ is the reduced state of the system. Since $\rho^{\mathcal{S}}(t)$ is always diagonalizable, there exists unitary transformation $U(t)$ which leads to

$$U^{\dagger}(t)\rho^{\mathcal{S}}(t)U(t) = \Lambda(t), \qquad (2)$$

where $\Lambda(t)$ is the diagonalized matrix whose diagonal elements are the eigenvalues of $\rho^{\mathcal{S}}(t)$. From (2) we can get

$$\rho^{\mathcal{S}}(t) = U(t)\Lambda(t)U^{\dagger}(t). \qquad (3)$$

Substituting (3) into (1) yields

$$\begin{aligned} S(t) &= -\mathrm{Tr}_{\mathcal{S}}\{U(t)\Lambda(t)U^{\dagger}(t)\ln[U(t)\Lambda(t)U^{\dagger}(t)]\} \\ &= -\mathrm{Tr}_{\mathcal{S}}\{U(t)\Lambda(t)U^{\dagger}(t)U(t)[\ln\Lambda(t)]U^{\dagger}(t)\} \\ &= -\mathrm{Tr}_{\mathcal{S}}[\Lambda(t)\ln\Lambda(t)] = -\sum_{j=1}^{n}\lambda_j(t)\ln\lambda_j(t), \end{aligned} \qquad (4)$$

where $\lambda_j(t)$ are eigenvalues of $\rho^{\mathcal{S}}(t)$, and $n$ is the dimension of the system. Here for simplicity we only consider finite dimensional quantum system. In order to control the entropy, we need to find the relationship between the controller and the time derivative of the entropy. The accurate derivative can be calculated using the sensitivity of eigenvalue. The first order derivative of the entropy can be calculated as

$$\dot{S}(t) = -\sum_{j=1}^{n}[1+\ln\lambda_j(t)]\dot{\lambda}_j(t). \qquad (5)$$

Assuming the right eigenvector of $\rho^{\mathcal{S}}(t)$ with respect to $\lambda_j(t)$ is $x_j(t)$, we can get

$$\rho^{\mathcal{S}}(t)x_j(t) = \lambda_j(t)x_j(t). \qquad (6)$$

Here the eigenvectors are assumed to be normalized, which means

$$x_j^{\dagger}(t)x_j(t) = 1. \qquad (7)$$

Since $\rho^{\mathcal{S}}(t)$ is Hermitian, the left eigenvector of $\rho^{\mathcal{S}}(t)$ with respect to $\lambda_j(t)$ should be the same as the right eigenvector, which means

$$x_j^{\dagger}(t)\rho^{\mathcal{S}}(t) = \lambda_j(t)x_j^{\dagger}(t). \qquad (8)$$

From (6) and (7) we can get

$$\lambda_j(t) = x_j^{\dagger}(t)\lambda_j(t)x_j(t) = x_j^{\dagger}(t)\rho^{\mathcal{S}}(t)x_j(t). \qquad (9)$$

From (9) the derivative of $\lambda_j(t)$ can be calculated as

$$\begin{aligned} \dot{\lambda}_j(t) &= \dot{x}_j^{\dagger}(t)\rho^{\mathcal{S}}(t)x_j(t) + x_j^{\dagger}(t)\dot{\rho}^{\mathcal{S}}(t)x_j(t) \\ &\quad + x_j^{\dagger}(t)\rho^{\mathcal{S}}(t)\dot{x}_j(t). \end{aligned} \qquad (10)$$

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Substituting (6), (7) and (8) into (10) yields

$$\begin{aligned}\dot{\lambda}_j(t) &= \dot{x}_j^\dagger(t)\lambda_j(t)x_j(t) + x_j^\dagger(t)\dot{\rho}^{\mathcal{S}}(t)x_j(t) \\ &\quad + \lambda_j(t)x_j^\dagger(t)\dot{x}_j(t) \\ &= x_j^\dagger(t)\dot{\rho}^{\mathcal{S}}(t)x_j(t) \\ &\quad + \lambda_j(t)[\dot{x}_j^\dagger(t)x_j(t) + x_j^\dagger(t)\dot{x}_j(t)] \\ &= x_j^\dagger(t)\dot{\rho}^{\mathcal{S}}(t)x_j(t) + \lambda_j(t)\frac{\mathrm{d}}{\mathrm{d}t}[x_j^\dagger(t)x_j(t)] \\ &= x_j^\dagger(t)\dot{\rho}^{\mathcal{S}}(t)x_j(t). \quad (11)\end{aligned}$$

Substituting (11) into (5) we can get

$$\dot{S}(t) = -\sum_{j=1}^{n}[1+\ln\lambda_j(t)]x_j^\dagger(t)\dot{\rho}^{\mathcal{S}}(t)x_j(t).$$

This is the closed form of the accurate time derivative of von Neumann entropy. We can use this to accurately calculate the derivative of entropy at any time. Although it needs to calculate all the eigenvalues and eigenvectors of $\rho^{\mathcal{S}}(t)$, it can be done by computer easily. What's more, for quantum system $\dot{\rho}^{\mathcal{SE}} = -i[H_{\text{tot}}, \rho^{\mathcal{SE}}]$ with $H_{\text{tot}} = H^{\mathcal{S}} \otimes I^{\mathcal{E}} + I^{\mathcal{S}} \otimes H^{\mathcal{E}} + H_{\text{int}}$, the derivative of entropy can similarly be written as

$\dot{S}(t) = i\sum_j [1+\ln\lambda_j(t)]x_j^\dagger(t)[H_{\text{int}}(t), \rho^{\mathcal{SE}}(t)]x_j(t)$. Here $H^{\mathcal{S}} \otimes I^{\mathcal{E}}$ and $I^{\mathcal{S}} \otimes H^{\mathcal{E}}$ can not lead to any change in the entropy.

Our control goal is to drive the entropy to track a desired trajectory, for example, to decrease to zero along a straight line. We can control the entropy by controlling the time derivative of the entropy. In Section 3 we provide the controller design method based on time-dependent decay rate. If the decay rate can change with time and can be used as the controller in the future, we can use it to drive the entropy easily.

## III. CONTROLLER DESIGN METHOD FOR TIME-DEPENDENT DECAY RATE

Since for closed quantum systems, the entropy can not be changed, we consider the following master equation for open quantum systems

$$\dot{\rho} = -i[H,\rho] + \mathcal{L}(\rho), \quad (12)$$

where $\mathcal{L}(\rho)$ is the Lindblad generator. From (12) we can get the derivative of entropy

$$\dot{S}(t) = -\sum_{j=1}^{n}[1+\ln\lambda_j(t)]x_j^\dagger(t)\mathcal{L}(\rho)x_j(t). \quad (13)$$

Here $[H,\rho]$ can not directly change the entropy, but the external fields contained in $H$ can change $\mathcal{L}(\rho)$,thus change the entropy. The non-Markovian master equation with time-dependent decay rate is

$$\dot{\rho} = -i[H,\rho] + \sum_j \gamma_j(t)\left(V_j\rho V_j^\dagger - \frac{1}{2}\{V_j^\dagger V_j, \rho\}\right), \quad (14)$$

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where the decay rate $\gamma_j(t)$ can be time-dependent, and can be negative for non-Markovian quantum systems [12]. When $\gamma(t)$ is negative, it means the information flow goes from the environment to the system, thus the entropy of the system will decrease, and the system can be purified. If the decay rate can be used as the controller in the future, the entropy control problem can be simplified. So in this section we use the decay rate as the controller, and choose $H = H_0 + H_1 u(t) \equiv H_0$, which means the external field $u(t)$ is always set as zero. Since the control problem with one decay rate is more difficult than those with multiple decay rates, we only consider the equation with one rate

$$\dot{\rho} = -i[H,\rho] + \gamma(t)\left(V\rho V^{\dagger} - \frac{1}{2}\{V^{\dagger}V,\rho\}\right). \quad (15)$$

From (12), (13) and (15) we can get

$$\begin{aligned}\dot{S}(t) = -\sum_{j=1}^{n}[1+\ln\lambda_j(t)]x_j^{\dagger}(t)\gamma(t)\times \\ \left(V\rho V^{\dagger} - \frac{1}{2}\{V^{\dagger}V,\rho\}\right)x_j(t).\end{aligned} \quad (16)$$

It is clear that

$$\begin{aligned}x_j^{\dagger}(t)V^{\dagger}V\rho x_j(t) &= x_j^{\dagger}(t)\rho V^{\dagger}Vx_j(t) \\ &= \lambda_j(t)x_j^{\dagger}(t)V^{\dagger}Vx_j(t).\end{aligned} \quad (17)$$

Substituting (17) into (16) leads to

$$\begin{aligned}\dot{S}(t) = \gamma(t)\sum_{j=1}^{n}[1+\ln\lambda_j(t)]x_j^{\dagger}(t)\times \\ [\lambda_j(t)V^{\dagger}V - V\rho V^{\dagger}]x_j(t).\end{aligned} \quad (18)$$

Define

$$\begin{aligned}\alpha(t) \triangleq \sum_{j=1}^{n}[1+\ln\lambda_j(t)]x_j^{\dagger}(t)\times \\ [\lambda_j(t)V^{\dagger}V - V\rho V^{\dagger}]x_j(t),\end{aligned} \quad (19)$$

and we can get

$$\dot{S}(t) = \gamma(t)\alpha(t). \quad (20)$$

So the controller can be calculated as

$$\begin{aligned}\gamma(t) &= \frac{\dot{S}(t)}{\alpha(t)} \\ &= \frac{\dot{S}(t)}{\sum_{j=1}^{n}[1+\ln\lambda_j(t)]x_j^{\dagger}(t)[\lambda_j(t)V^{\dagger}V - V\rho V^{\dagger}]x_j(t)}.\end{aligned} \quad (21)$$

This is the desired controller which can accurately drive the entropy to track a pre-specified trajectory $S(t)$. Here we use open loop control since the entropy is difficult to measure. We can substitute (21) into (15) to solve the evolution of the system, based on which the evolutions of both the entropy and the controller can be

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calculated. This controller can only take effect when $\lambda_j(t) \neq 0$ because the definition of $\ln 0$ doesn't exist. What's more, when $\alpha(t) = 0$, there will be singularity since there is division-by-zero problem in the controller. We can use the gradient method to solve the singularity problem. If we consider the minimization of the tracking error

$$\min J = [\dot{S}(t) - \gamma(t)\alpha(t)]^2 , \quad (22)$$

the controller can be calculated from

$$\dot{\gamma}(t) = -k\frac{\partial J}{\partial \gamma(t)} = 2k[\dot{S}(t) - \gamma(t)\alpha(t)]\alpha(t) , \quad (23)$$

where $k > 0$. This method can only avoid the division-by-zero problem at certain time, while cannot always guarantee minimization.

We can do simulation on 2-level quantum systems. When $H_0 = V = \sigma_z$, the evolution of the system is

$$\begin{bmatrix} \dot{\rho}_{11}(t) & \dot{\rho}_{12}(t) \\ \dot{\rho}_{21}(t) & \dot{\rho}_{22}(t) \end{bmatrix} = \begin{bmatrix} 0 & -2[i+\gamma(t)]\rho_{12}(t) \\ 2[i-\gamma(t)]\rho_{21}(t) & 0 \end{bmatrix} . \quad (24)$$

We can assume

$$\rho_{12}(t) = a(t) + ib(t) , \quad (25)$$

where $a(t), b(t) \in \mathbf{R}$. Substituting (25) into (24) leads to

$$\begin{cases} \dot{a}(t) = 2b(t) - 2\gamma(t)a(t) \\ \dot{b}(t) = -2a(t) - 2\gamma(t)b(t) . \end{cases} \quad (26)$$

For initial state

$$\begin{bmatrix} \rho_{11}(0) & \rho_{12}(0) \\ \rho_{21}(0) & \rho_{22}(0) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 0.5 \end{bmatrix} , \quad (27)$$

the state evolution at time *t* is

$$\begin{bmatrix} \rho_{11}(t) & \rho_{12}(t) \\ \rho_{21}(t) & \rho_{22}(t) \end{bmatrix} = \begin{bmatrix} 0.5 & a(t)+ib(t) \\ a(t)-ib(t) & 0.5 \end{bmatrix} . \quad (28)$$

Usually it is easier to make the entropy increase, while it is difficult to make it decrease. Here we choose $\dot{S}(t) = -1$, which means the entropy is desired to decrease along a straight line with decreasing scope $-1$. Substituting (28) into (21) we can get the controller

$$\gamma(t) = \frac{-1}{\sum_{j=1}^{2} [1+\ln \lambda_j(t)] x_j^\dagger(t) \begin{bmatrix} \lambda_j(t) - 0.5 & a(t)+ib(t) \\ a(t)-ib(t) & \lambda_j(t) - 0.5 \end{bmatrix} x_j(t)} . \quad (29)$$

Substituting (29) into (26) we can get the evolution of the system

$$\begin{cases} \dot{a}(t) = 2b(t) - \\ \dfrac{2\dot{S}(t)a(t)}{\sum_{j=1}^{2}[1+\ln\lambda_j(t)]x_j^{\dagger}(t)\begin{bmatrix}\lambda_j(t)-0.5 & a(t)+ib(t)\\ a(t)-ib(t) & \lambda_j(t)-0.5\end{bmatrix}x_j(t)} \\ \dot{b}(t) = -2a(t) - \\ \dfrac{2\dot{S}(t)b(t)}{\sum_{j=1}^{2}[1+\ln\lambda_j(t)]x_j^{\dagger}(t)\begin{bmatrix}\lambda_j(t)-0.5 & a(t)+ib(t)\\ a(t)-ib(t) & \lambda_j(t)-0.5\end{bmatrix}x_j(t)}, \end{cases} \quad (30)$$

where $a(0)=0.3$, $b(0)=0$. The simulation of (30) can be done in MATLAB using discretized approximation, and the evolutions of both the entropy and the controller are shown in Fig. 1 and Fig. 2 respectively. Here the sampling period is chosen as $T=0.0001$, and the eigenvalues and eigenvectors can be directly simulated by the eig function in MATLAB.

From Fig. 1 we can see that the entropy can decrease to zero along a straight line. Such accurate tracking method can also be used to drive the entropy to track any pre-specified trajectory.

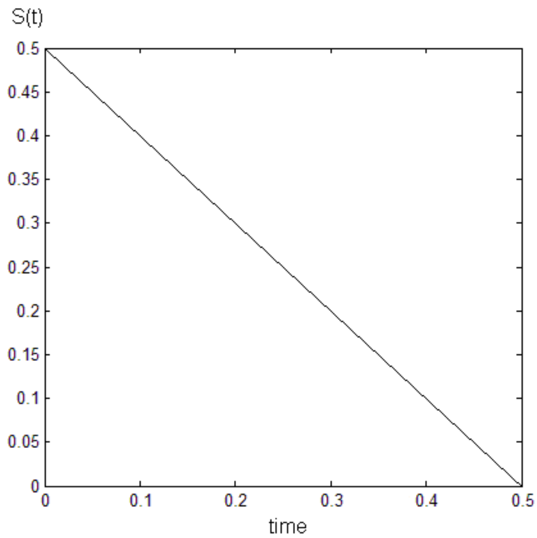


**Figure 1:** Time evolution of von Neumann entropy for system (24) with initial state (27) under controller (29).

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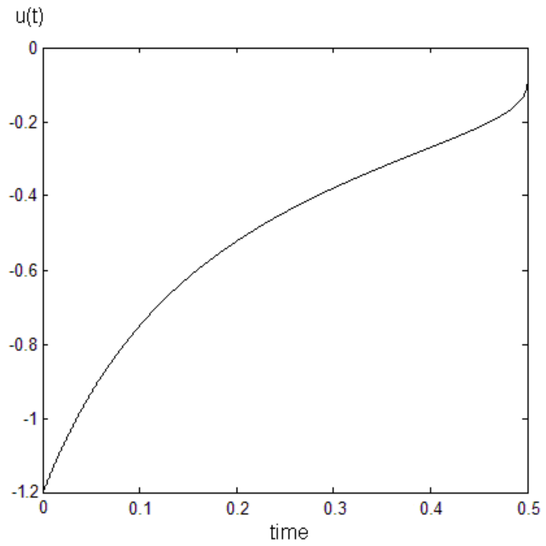


**Figure 2:** Time evolution of the entropy controller (29) for system (24) with initial state (27).

## IV. CONCLUSION

This paper proposes a new quantum control method which accurately controls the von Neumann entropy of quantum systems. Simulation example evidenced the effectiveness of the method. Our methods provide a universal tool for entropy control, which can also contribute to classical information theory. The extension of the controller to other types of entropy deserves our future research. The applications of the entropy controllers in negative entropy [13,14] and information flow [15-18] are also of keen interests and currently being pursued.

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